Embedding of Cycles into Hypercube

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Abstract- We present an approach to find the edge congestion sum and dilation sum forembedding of cycle on n vertices, into hypercube.
Keywords –Embedding, Hypercube, Cycles, Congestion Sum, Dilation Sum

1. INTRODUCTION

Definition: A **graph** is a pair where the set of all vertices in *G* and *E* is is the set of all edges in *G*. We call a graph *G(V,E)* is finite if *V* and *E* both are finite.
Definition: Let *G(V,E)* and) be two finite graphs. A 1-1 mapping is called an **embedding**. Graph is called a **host graph** and graph is called **guest or virtual graph.**
The Dilation Problem
Definition: Let *G(V,E)* and) be two finite graphs. Let be an embedding of into Then the dilation of into with respect to ,denoted by , is defined as
$D_f(G,H) = \max\{d_H(f(u),f(v)) \mid (u,v)\epsilon E\}$,
where denotes the length of the shortest path between and in .
Definition: The dilation of in to is denoted by , is defined as where the minimum is taken over all embedding of *G* in to *H*.
Definition: The dilation problem is to find an embedding of onto that gives minimum dilation.
The Dilation Sum Problem
Definition: The dilation sum of an embedding f of into is denoted by and is defined as
=
Definition: The dilation sum of *G* into *H*, denoted by *D'(G,H)*, is defined as *D'(G,H)*=min
where the minimum is taken over all embeddings of *G* into *H*.
The Congestion Sum Problem
Definition: The **congestion** of an embedding is the maximum number of edges of the guest graph that are embedded to any single edge of the host graph. For an embedding f, of G in to H, let there is a unique path, for every edge in , in from to . Let denotes this path and denotes congestion on the edge in .Then
$C_f(G,H(e)) = |\{(u,v)\epsilon E(G) \mid e\epsilon P_f(f(u),f(v))\}|$
Definition: Let be an arbitrary embedding of in to . Then the **congestion sum** of is defined as

The minimum congestion sum of in to is defined as
=min.
where the minimum is taken over all embedding of into .
Definition: The **congestion sum problem** is to find an embedding of on to that gives minimum congestion sum.
We shall denote by .
REMARK: The congestion sum problem and dilation sum problem are same.

II. OVERVIEW OF THE ARTICLE

The dilation-sum problem has been studied for binary trees into paths [8, 12], hypercubes into grids [5], complete graphs into hypercubes [19]. The bounded cost of dilation and congestion has been estimated for the embedding on binary trees [27]. Most of the work on the dilation-sum problem and the dilation problem are for the particular case in which the host graph is a path, or a cycle [20].The concept of cutwidth is a special case of congestion when the host graph is a path [11, 26, 29]. There are several results on the congestion problem for various architectures such as trees into cycles [11], trees into stars [28], trees into hypercubes [4, 22], hypercubes into grids [5, 6, 25], complete binary trees into grids [23], and ladders and caterpillars into hypercubes [7, 10]. There are also other general results on embeddings [2]. There are algorithms for the embedding of Cycles and wheel into arbitrary tree [30] and k sequential m –ary into hypercube[31]. In this article we produce an embedding which gives us minimum dilation using the graph isomorphism and properties of hypercube.

III.EMBEDDING OF CYCLES INTO HYPERCUBE

Definition: An n-dimensional hyper cube is a graph with vertices represented by all binary n-tuple and two vertices are adjacent if and only if their corresponding r- tuples differ in exactly one position.

The decimal representation of the vertices is , but for convenience we will take the symbol for and so the set of labels of vertices is .

Definition: Let be an n-dimensional hypercube. A partial ordering " " on is defined by if and only if is a subcube of , 1 i, j . The notation < shall mean < and .

Definition: A hamiltonianlabelling of hypercube ,denoted by **hal**, is the labelling of the vertices of defined inductively as follows: Consider the ordering on . Label vertices of as 0 and 1. If is labelled , then label the unique vertex adjacent to u as

Definition : Ahamiltonian cycle is a cycle that visits each node of the graph exactly once. By convention, the trivial graph on a single node is considered to possess a hamiltonian cycle. A graph possessing a Hamiltonian cycle is said to be a hamiltonian graph.

Theorem 1 The Hamiltonian labellinghal of determines a hamiltonian cycle in , .

Proof We prove that haldetermines the hamiltonian cycle in for all .We prove the result by induction on r. For r = 2, is a hamiltonian cycle in . Assume that is a hamiltonian cycle in .Consider . We observe that is a subcube of . Let denote the other – dimensional subcube contained in By definition of , is an edge in if and only if is an edge in , where and are edges in . Thus is a hamiltonian path in implying that $P' = ((2^k \oplus 1)$ $1=$ is a hamiltonian path in . Moreover, the vertex labeled is adjacent to the vertex labelled = . Again the vertex labeled 1 is adjacent to the vertex labeled . Thus $P \circ P^{r-1} \circ (2^k, 1)$ is a hamiltonian cycle in .

Theorem2: For every even integer such that , ( )contains a cycle of length

Proof: We will prove this result by using induction on . For , contains cycle of length four and contains cycles of length 4,6 and 8. Let us assume that contains cycles of lengths 4,6,8…, . Consider . We observe that is a sub-cube of . Let denote the other –dimensional sub-cube contained in . Now for every even integer , if then has a cycle of length and so does . Now if , then can be written where is an even integer . So by inductionhypothesis has a cycle of length named (say) and has a cycle of length named (say). By definition of , is an edge in if and only if is an edge in , where and are edges in . Using this argument is an edge in if and only if is an edge in , where and are edges in Now removing the edge from and from and adding edges and we get a cycle in of length This completes the induction.

Definition: Let be a graph on vertices then the hypercube of dimension is called optimal hypercube.

Theorem 3: Let and are two finite graphs on n vertices and are isomorphic. Then the dilation of embedding of into is 1 and the dilation sum is .

Proof: The proof follows from the definition of dilation and isomorphism.

Theorem4: The dilation of embedding of , the cycle graph on n vertices , into the optimal hypercube is and the dilation sum and congestion sum is

Proof: When is even. contains a cycle of length . Thus is isomorphic to a sub-graph of . The dilation of this isomorphism is 1 and the dilation sum is .

When is odd. We embed in to the cycle in , . Under this mapping the dilation of each edge except is 1 and the dilation on is 2. Since the dilation of an embedding is always greater than or equal to 1. But dilation is equal to 1 iff when the guest graph is isomorphic to the host graph. But in this case no isomorphism is possible since there are no odd cycles in the hypercube. Thus this mapping is an embedding with dilation 2 and the dilation sum is . Since the dilation sum and congestion sum are same so the congestion sum of this embedding is

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