Unique Fixed Point Theorems In Metric Space

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Abstract - In this paper, we have established unique fixed point theorems in complete metric space and generalized in n-dimensional space.

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Key words: Complete metric space, contraction mapping, Banach fixed point theorem, continuous mapping

I. INTRODUCTION

Fixed point theory plays a basic role in applications of many branches of mathematics and other many areas. The problem of finding fixed point has been studied in several directions. The study of metric fixed point theory has been researched extensively in past decades. Recently, some generalizations of the notion of a metric space have been proposed by many authors and finding a fixed point of contractive mapping in complete metric space. There are many works concerning the fixed point of contractive maps $(see, for \quad example, [10,13])$. In $[10]$, Polish mathematician Banach (1922) proved a very important result regarding a contraction mapping, known as the Banach contraction principle . It is also known as Banach fixed point theorem.
The aim of this paper, is to prove some fixed point theorems, which based on contraction mappings, Banach contraction principle, and focused on the some results of $[1-16]$.

1.1 Continuous mapping : A self map T on a Banach space X is said to be continuous at a point $x \in X$ if and only if

$$x_n \to x \Rightarrow T(x_n) \to T(x).$$

1.2 Complete metric space : A metric space (X,d) is said to be complete if every Cauchy sequence in X, converges in X.

1.3 Contraction mapping : A self map T defined on a metric space (X,d) is said to be a contraction , if satisfying

$$d(Tx,Ty) \le \alpha d(x,y)$$

for all $x, y \in X$, $x \neq y$ and $0 < \alpha < 1$.

1.4 Banach fixed point theorem : Let (X,d) be a complete metric space and $T : X \to X$ be a contraction on X. Then T has a unique fixed point in X.

II. MAIN RESULTS

Theorem 2.1: Let T be a continuous self map defined in a complete metric space (X,d), satisfying the condition

$$d(Tx,Ty) \le \alpha d(x,Tx) + \beta d(y,Ty) + \gamma d(x,y) \tag{1}$$

for all $x, y \in X$, $x \ne y$ and for some $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$, then T has a unique fixed point in X.

Proof : Let x_0 be an arbitrary point in X and $\{x_n\}_{n=0}^{\infty}$ be the sequence of iterations of T at x_0, i.e. $x_{n+1} = T(x_n)$ $\forall n \in N$. If $x_n = x_{n+1}$ for some n then the result follows trivially . So, let $x_n \ne x_{n+1}$ for all n.

Obviously, we have

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})).$$

Using (1) in above, we get

$$d(x_{n+1}, x_n) \le \alpha d(x_n, T(x_n)) + \beta d(x_{n-1}, T(x_{n-1})) + \gamma d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \le \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n-1})$$

$$\Rightarrow d(x_{n+1}, x_n) \le (\frac{\beta + \gamma}{1 - \alpha}) d(x_n, x_{n-1}).$$

Similarly, proceeding this work, we get

$$\Rightarrow d(x_{n+1}, x_n) \le (\frac{\beta + \gamma}{1 - \alpha})^n d(x_1, x_0).$$

By the triangle inequality, we have for $m \ge n$

$$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m)$$

$$\le (K^n + K^{n+1} + ... + K^{m-1}) d(x_0, x_1)$$

where $K = (\frac{\beta + \gamma}{1 - \alpha})$

$$= (\frac{K^n}{1 - K}) d(x_0, x_1) \ as\, m \to \infty$$

$$\to 0 \ as\, n \to \infty.$$

Since X is complete, there exist a point $p \in X$ such that $x_n \to p$. Further, the continuity of T in X, implies that

$$T(p) = T(\lim_{n \to \infty} x_n)$$

$$= \lim_{n \to \infty} T(x_n) = p$$

Therefore p is a fixed point of T in X.

Now, we prove that p is a unique fixed point of T. If there exist another fixed point $q (\ne p) \in X$, then we have

$$d(p,q) = d(T(p), T(q))$$

$$\le \alpha d(p, T(p)) + \beta d(q, T(q)) + \gamma d(p,q)$$

$$\le \alpha d(p,p) + \beta d(q,q) + \gamma d(p,q)$$

$$= \gamma d(p,q)$$

$$\Rightarrow d(p,q) < d(p,q) \ (since\ \gamma < 1),$$

which is a contradiction. Hence p is a unique fixed point of T in X.

Theorem 2.2: Let T be a self map defined on a complete metric space (X,d), which satisfying the condition (1). If for some positive integer r, T^r is continuous, then T has a unique fixed point.

Proof : Consider a sequence $\{x_n\}$ as in theorem (2.1). It converges to some point $p \in X$. Therefore, its subsequence $\{x_{n_k}\}$ also converges to p. Also,

$$T^r(p) = T^r(\lim_{k\to\infty} x_{n_k})$$

$$= \lim_{k\to\infty} x_{n_{k+r}} = p$$

Therefore p is a fixed point of T^r. Now, we have to show that p is a fixed point of T, i.e., $T(p) = p$. Let m be the smallest positive integer such that $T^m(u) = u$ but $T^l(p) \neq p$ for $l = 1,2,3,\cdots,m-1$. If $m > 1$, then we have

$$d(T(p),p) = d(T(p),T^m(p)) = d(T(p),T(T^{m-1}(p)))$$

$$\leq \alpha d(p,T(p)) + \beta d(T^{m-1}(p),T^m(p)) + \gamma d(p,T^{m-1}(p))$$

$$= \alpha d(p,T(p)) + \beta d(T^{m-1}(p),T(p)) + \gamma d(p,T^{m-1}(p))$$

$$\Rightarrow d(T(p),p) = (\frac{\beta+\gamma}{1-\alpha}) d(T^{m-1}(p),p). \tag{2}$$

Again, we have

$$d(p,T^{m-1}(p)) = d(T^m(p),T^{m-1}(p))$$

$$\leq \alpha d(T^{m-1}(p),T^m(p)) + \beta d(T^{m-2}(p),T^{m-1}(p)) + \gamma d(T^{m-1}(p),T^{m-2}(p))$$

implying that

$$d(p,T^{m-1}(p)) \leq (\frac{\beta+\gamma}{1-\alpha}) d(T^{m-2}(p),T^{m-1}(p)).$$

Similarly, proceeding this work, we obtain

$$d(p,T^{m-1}(p)) \leq (\frac{\beta+\gamma}{1-\alpha})^{m-1} d(T(p),p). \tag{3}$$

From (2) and (3), we have

$$d(T(p),p) \leq (\frac{\beta+\gamma}{1-\alpha})^m d(T(p),p)$$

since $\alpha+\beta+\gamma<1$

$$< d(T(p),p),$$

which is a contradiction. Hence $T(p) = p$, i.e. p is a fixed point of T. We have to show that, the fixed point of T is unique. suppose $q(\neq p) \in X$ be another fixed point of T, then we have

$$d(q,p) = d(T(q),T(p))$$

$$\leq \alpha d(q,T(q)) + \beta d(p,T(p)) + \gamma d(p,q)$$

$$\leq \gamma d(p,q)$$

$$< d(p,q)\ (since\, \gamma < 1),$$

which is a contradiction. Hence fixed point of T is unique.
In the next theorem, we generalized the theorem (2.1) and (2.2).

Theorem 2.3: Let T be a self map defined on a complete metric space (X,d) and for some positive integer $'t'$, satisfying the condition

$$d(T^t(x),T^t(y)) \leq \alpha d(x,T^t(x)) + \beta d(y,T^t(y)) + \gamma d(x,y) \tag{4}$$

for all $x,y \in X, x \neq y$ and for some $\alpha,\beta,\gamma \in [0,1)$ with $\alpha+\beta+\gamma<1$. If T^t is continuous, then T has a unique fixed point.
Proof : The proof of theorem (2.3) is similar to the theorem (2.1) and (2.2).

In the next theorem, we will study the existence of a unique common fixed point of two mappings which are not necessarily continuous or commuting.

Theorem 2.4: Let T_1 and T_2 be two self mappings defined on a complete metric space (X,d), satisfying the following conditions

(i) For some $\alpha,\beta,\gamma \in [0,1)$ with $\alpha+\beta+\gamma<1$ such that

$$d(T_1(x),T_2(y)) \leq \alpha d(x,T_1(x)) + \beta d(y,T_2(y)) + \gamma d(x,y) \tag{5}$$

for all $x,y \in X, x \neq y.$

(ii) T_1T_2 is continuous on X.

(iii) There exist an $x_0 \in X$ and the sequence $\{x_n\}$,

$$where \quad x_n = \begin{cases} T_1(x_{n-1}) : whennisodd \\ T_2(x_{n-1}) : whenniseven \end{cases}$$

such that $x_n \neq x_{n+1}$ for all n. Then T_1 and T_2 having a unique common fixed point.
Proof : Consider,

$$d(x_{2n+1},x_{2n}) = d(T_1x_{2n},T_2x_{2n-1})$$

using (5) in above equation, we have

$$\leq \alpha d(x_{2n},T_1x_{2n}) + \beta d(x_{2n-1},T_2x_{2n-1}) + \gamma d(x_{2n},x_{2n-1})$$

$$\Rightarrow d(x_{2n+1},x_{2n}) \leq (\frac{\beta+\gamma}{1-\alpha}) d(x_{2n},x_{2n-1})$$

continuing this work, we have

$$\leq (\frac{\beta+\gamma}{1-\alpha})^{2n} d(x_1,x_0).$$

Similarly,

$$d(x_{2n+2},x_{2n+1}) \leq (\frac{\beta+\gamma}{1-\alpha})^{2n+1} d(x_1,x_0).$$

Now, it can be easily seen that $\{x_n\}$ is a cauchy sequence. Let $\{x_n\} \to p$, then the subsequence

$\{x_{n_k}\} \to p$. Then, we have

$$T_1T_2(p) = T_1T_2(lim_{k\to\infty}x_{n_k}) = lim_{k\to\infty}x_{n_{k+1}} = p.$$

i.e. p is a fixed point of T_1T_2. Next, we have to show that $T_2(p) = p$. Suppose $T_2(p) \neq p$, then

$$d(T_2(p), p) = d(T_2(p), T_1T_2(p))$$

$$\leq \alpha d(p, T_2(p)) + \beta d(T_2(p), T_1T_2(p)) + \gamma d(p, T_2(p))$$

$$= (\alpha + \beta + \gamma) d(p, T_2(p))$$

since $\alpha + \beta + \gamma < 1$

$$< d(p, T_2(p)),$$

which is a contradiction. Hence $T_2(p) = p$, i.e. p is a fixed point of T_2.

Also,

$$d(T_1(p), p) = d(T_1(T_2(p)), p) = d(T_1T_2(p), p) = d(p, p) = 0.$$

Hence $T_1(p) = p$, i.e. p is a fixed point of T_1. So T_1 and T_2 have a common fixed point. Now, we have to prove that T_1 and T_2 have a unique common fixed point. If possible, let $q(\neq p) \in X$ be two fixed points of T_1 and T_2. Then

$$d(p, q) = d(T_1(p), T_2(q))$$

$$\leq \alpha d(p, T_1(p)) + \beta d(q, T_2(q)) + \gamma d(p, q)$$

$$= \alpha d(p, p) + \beta d(q, q) + \gamma d(p, q)$$

$$= \gamma d(p, q)$$

$$< d(p, q) \ (since\, \gamma < 1),$$

which is a contradiction. Hence p is a unique common fixed point of T_1 and T_2.

Remark 2.1: If $X = [0,1]$ and T_1, T_2 be two commuting continuous self mappings defined on X, then T_1 and T_2 need not have a common fixed point by Smart [6].

Theorem 2.5: Let T_n be a self map defined on a complete metric space (X, d) with p_n as a fixed point for each $n = 1, 2, 3, \cdots$ respectively. If T_n satisfying the condition (1) for all n and $\{T_n\}$ converges point wise to T, then $p_n \to p$ as $n \to \infty$ if and only if p is a fixed point of T.

Proof : If $p = p_n$ for some n, then the assumption follows trivially. So, we assume that $p \neq p_n$ for any n. Let $p_n \to p$ then, we show that p is fixed point of T. Now,

$$d(p, T(p)) \leq d(p, p_n) + d(p_n, T_n(p)) + d(T_n(p), T(p))$$

$$= d(p, p_n) + d(T_n(p_n), T_n(p)) + d(T_n(p), T(p))$$

$$\leq d(p, p_n) + \alpha d(p_n, T_n(p_n)) + \beta d(p, T_n(p)) + \gamma d(p_n, p) + d(T_n(p), T(p)).$$

Since p_n is a fixed point of T_n and $T_n \to T$ as $n \to \infty$, then we have

$$(1-\beta)d(p,T(p)) \le 0$$

$$\Rightarrow d(p,T(p)) \le 0 \ (since\, \beta < 1)$$

$$\Rightarrow d(p,T(p)) = 0$$

$$\Rightarrow T(p) = p.$$

Conversely, we suppose that $T(p) = p.$ Then obviously, we have

$$d(p_n, p) = d(T_n(p_n), T(p))$$

$$\le d(T_n(p_n), T_n(p)) + d(T_n(p), T(p))$$

$$\le \alpha d(p_n, T_n(p_n)) + \beta d(p, T_n(p)) + \gamma d(p_n, p) + d(T_n(p), T(p))$$

$$\Rightarrow d(p_n, p) \le (\frac{1+\beta}{1-\gamma}) d(p, T_n(p))$$

$$= (\frac{\beta+1}{1-\gamma}) d(T(p), T_n(p))$$

$$\to 0 \ as\, n \to \infty, T_n \to T.$$

Hence $p_n \to p$.

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