

# Theory of Generalized Backward Difference Operator and its Applications in Numerical Methods

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**Abstract -** In this paper, we define the generalized backward difference operator and obtain its relation with usual shift operator. Also, we present the discrete version of Leibnitz theorem according to the generalized backward difference operator. By defining its inverse, and using Stirling numbers of first kind and second kind, we establish a formula for sum higher powers of arithmetic progression in the field of Numerical Methods.

**Key words :** Generalized Difference Operator, Strirling Numbers, Arithmetic Progression.

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## I. INTRODUCTION

The theory of difference equations is developed with the definition of the difference operator

$$\nabla y_k = y_k - y_{k-1}, k \in N \quad (1)$$

where  $N$  is the set of natural numbers. The definition of  $\nabla$  is simply the difference between two successive values of the sequence  $\{y_k\}$ ,  $k \in N$ . Many authors [see 1-4] suggested the possible study by redefining the operator as

$$\nabla_\ell y_k = y_k - y_{k-\ell}, \ell \in N. \quad (2)$$

But no significant progress took place on this line. The theory developed already with the difference operator  $\nabla$  agrees when  $\ell = 1$ . In this paper, we develop the basic theory for the generalized difference operator  $\nabla_\ell$  and obtain relations connecting  $\nabla_\ell$  and  $\nabla$ ,  $\nabla_\ell$  and  $E$  and the basic properties of  $\nabla_\ell$  and also obtain a formula for finding the sum of the  $n^{\text{th}}$  powers of an arithmetic progression and for an arithmetico-geometric progression by defining  $\nabla_\ell^{-1}$ . In Section 2, we define the generalised difference operator  $\Delta'$  and establish relationships of  $\Delta'$  with  $\Delta$  and  $E$  respectively. Basic properties of  $\Delta'$  such as product and quotient rules of  $\Delta'$  are obtained. In Section 3, we define the higher orders of  $\Delta'$  and obtain the generalised version of Libnitz theorem and Binomial theorem. In Section 4, we define the generalised factorial and obtain generalised version of the Newton's formula for a polynomial  $f(k)$ .

Finally, in Section 5, we define the inverse of the generalised operator  $\Delta$  and obtain formulae for finding the sum of the  $n$ th powers of an arithmetic and arithmetic-geometric Progressions respectively using Stirling numbers of second kind. Suitable examples are presented to establish the results. The formulae obtained are verified by using c-programming and tested its validity.

II. GENERALIZED BACKWARD DIFFERENCE OPERATOR AND ITS RELATION WITH E

**Definition 2.1.** Let  $u(k)$  be real valued function on  $[0, \infty)$ . Then, we define the generalized backward difference operator  $\nabla_\ell$  is defined as

$$\nabla_\ell u(k) = u(k) - u(k - \ell) \tag{3}$$

**Lemma 2.2.** The relation between  $\nabla_\ell$  and E is

$$E^{-\ell} = 1 - \nabla_\ell = (1 - \nabla)^\ell \tag{4}$$

**Lemma 2.3.** The relation between  $\nabla_\ell$  and powers of  $\nabla$  is

$$1 - \nabla_\ell = \sum_{i=0}^{\ell} (-1)^i \ell C_i \nabla^i \tag{5}$$

**Remark 2.7.** (i) If  $a, b$  are non-zero scalars and  $u(k)$  and  $v(k)$  are real valued functions then,

$$\nabla_\ell (au(k) + bv(k)) = a\nabla_\ell u(k) + b\nabla_\ell v(k) \tag{6} \text{ (ii)}$$

If  $\ell$  in a positive real and  $u(k), v(k)$  are any two real valued functions, then

$$\nabla_\ell (u(k)v(k)) = u(k)\nabla_\ell v(k) + v(k - \ell)\nabla_\ell u(k) \tag{7}$$

(iii) If  $\ell$  is a positive real and  $u(k), v(k)$  are any two real valued functions then

$$\nabla_\ell \left\{ \frac{u(k)}{v(k)} \right\} = \frac{v(k)\nabla_\ell u(k) - u(k)\nabla_\ell v(k)}{v(k)v(k - \ell)} \tag{8}$$

III. HIGHER ORDERS OF  $\nabla_\ell$

In this section, we define the higher orders of  $\nabla_\ell$  and establish the generalized version of Leibnitz theorem and Binomial theorem.

**Definition 3.1.** The generalized second order difference operator denoted by  $\nabla_\ell^2$  is defined as

$$\nabla_\ell^2 = \nabla_\ell (\nabla_\ell) \tag{9}$$

In general, the  $n^{th}$  order generalised backward difference operator denoted by  $\nabla_\ell^n$  is defined as

$$\nabla_\ell^n = \nabla_\ell (\nabla_\ell^{n-1}). \tag{10}$$

We present the following remarks which can be easily established.

**Lemma 3.2.** If  $l, m$  and  $n$  are any positive integers then

$$\nabla_\ell^n k^n = n! \ell^n \text{ and } \nabla_\ell^n k^m = 0 \text{ if } m > n. \tag{11}$$

*Proof.* The proof can be easily established by induction on n.

**Remark 3.3.** If  $p_k = a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \dots + a_n$  is any  $n^{th}$  degree polynomial in k then

$$\nabla_\ell^n p_k = a_0 \ell^n n! \quad \text{and} \quad \nabla_\ell^m p_k = 0 \quad \text{if} \quad m > n \quad (12)$$

**Lemma 3.4.** If  $u(k)$  is a real valued function, then

$$\nabla_\ell^r u(k) = \sum_{i=0}^r (-1)^{r-i} r C_i u(k - \ell(r-i)). \quad (13)$$

**Lemma 3.5.** If  $\ell$  is a positive real,  $k, r, n$  are positive integers then

$$\nabla_\ell^r (k^{(n)}) = \begin{cases} \sum_{i=0}^r (-1)^{r-i} r C_i (k - \ell(r-i))^n, & r < n \\ n! \ell^n, & r = n \\ 0, & r > n \end{cases} \quad (14)$$

**Lemma 3.6.** If  $\ell_1, \ell_2, \ell_3, \dots, \ell_n$  are positive reals, then

$$1 - \nabla_{\sum_{i=1}^n \ell_i} = \prod_{i=1}^n (1 - \nabla_{\ell_i}) \quad (15)$$

The discrete version of Leibnitz's theorem according to  $\nabla_\ell$  is given below.

**Theorem 3.7.** If  $u(k)$  and  $v(k)$  are two real valued functions, then

$$\nabla_\ell^n [u(k)v(k)] = \sum_{i=0}^n n C_r \nabla_\ell^i u(k) \nabla_\ell^{n-i} v(k - i\ell). \quad (16)$$

*Proof.* Define the operators  $E_1^{-\ell}$  and  $E_2^{-\ell}$  as

$$E_1^{-\ell} (u(k)v(k)) = u(k - \ell)v(k) \quad \text{and} \quad E_2^{-\ell} (u(k)v(k)) = u(k)v(k - \ell) \quad (17)$$

Hence, we get

$$E_1^{-\ell} E_2^{-\ell} = E^{-\ell} \quad (18)$$

Also, define  $(\nabla_\ell)_1 = 1 - E_1^{-\ell}$  and  $(\nabla_\ell)_2 = 1 - E_2^{-\ell}$  (19)

This implies  $\nabla_\ell = 1 - E^{-\ell} = 1 - E_1^{-\ell} E_2^{-\ell}$  (20)

From (18), we get  $\nabla_\ell = (\nabla_\ell)_2 + (\nabla_\ell)_1 E_2^{-\ell}$

Hence,  $\nabla_\ell^n [u(k)v(k)] = \{ (\nabla_\ell)_2 + (\nabla_\ell)_1 E_2^{-\ell} \}^n u(k)v(k)$

The proof follows using Binomial theorem (3.14) and (3.15).

**Lemma 3.8.** If  $n$  is positive integer and  $\ell$  is positive real, then

$$E^{-n\ell} = \sum_{r=0}^n (-1)^r n C_r \nabla_\ell^r \quad (21)$$

The following theorem is the generalized version of the Binomial theorem

**Theorem 3.9.** If  $m$  and  $n$  are any two positive integers, then

$$(k - n\ell)^m = \sum_{r=0}^n \sum_{i=0}^r (-1)^{r+i} n C_r r C_i (k - i\ell)^m. \tag{22}$$

*Proof.* The proof follows by operating (21) on  $u(k) = k^m$ .

**Lemma 3.10.** If  $u(k)$  is a real valued function  $\ell$  is positive real and  $x$  is real, then

$$\sum_{j=0}^{\infty} \frac{x^{j\ell} u(-j\ell)}{j! \ell^j} = \left\{ e^{\frac{x^\ell k^{-\ell}}{\ell}} \right\} u(0) = \left\{ e^{\frac{x^\ell}{\ell}} e^{-\frac{x^\ell \nabla_\ell}{\ell}} \right\} u(0) \tag{23}$$

**Example 3.19.** Sum to infinity series

$$1 - \frac{(1 - \ell^2) x^\ell}{\ell} + \frac{(1 - (2\ell)^2) x^{2\ell}}{2! \ell^2} + \frac{(1 - (3\ell)^2) x^{3\ell}}{3! \ell^3} + \dots + \infty$$

**Solution:** Here  $u(k) = (1 - k^2)$ ,  $u(j\ell) = (1 - (j\ell)^2)$

$$\nabla_\ell u(k) = \ell^2 - 2k\ell, \nabla_\ell^2 u(k) = -2\ell^2, \nabla_\ell^3 u(k) = 0$$

In general,  $\nabla_\ell^n u(k) = 0$  for  $n \geq 3$

Put  $k = 0$ , we have

$$\nabla_\ell u(0) = \ell^2, \nabla_\ell^2 u(0) = -2\ell^2, \nabla_\ell^3 u(0) = 0$$

Hence by (23), Sum  $= e^{\frac{x^\ell}{\ell}} \{1 - \ell x^\ell - x^{2\ell}\}$ .

#### IV. THE GENERALIZED POLYNOMIAL FACTORIAL

In this section, we define the generalised polynomial factorial and obtain the generalised version of the Newton's Backward formula for a polynomial  $f(k)$ .

**Definition 4.1.** If  $n$  is positive integer and  $\ell$  is positive real then, the generalized polynomial factorial denoted by  $k_\ell^{(n)}$  is defined as,

$$k_\ell^{(n)} = k(k - \ell)(k - 2\ell) \dots (k - (n - 1)\ell). \tag{24}$$

**Lemma 4.4.** If  $s_r^n$ 's are the Stirling numbers of first kind, then

$$k_\ell^{(n)} = \sum_{r=1}^n s_r^n \ell^{n-r} k^r. \tag{25}$$

**Lemma 4.5.** If  $S_r^n$ 's are the Stirling numbers of first kind, then

$$k^n = \sum_{r=1}^n S_r^n \ell^{n-r} k_\ell^{(r)}. \tag{26}$$

**Lemma 4.6.** If  $s_r^n$ 's are Stirling numbers of first kind, then

$$\nabla_{\ell}^m k_t^{(n)} = \sum_{r=1}^m (-1)^r m C_r \sum_{i=1}^n s_i^n (k-\ell)^r t^{n-r}$$

*Proof.* The proof follows from lemma 3.5 and (24).

**Lemma 4.7.** If  $\ell$  is positive real and  $n$  is positive integer, then

$$\nabla_{\ell} k_{\ell}^{(n)} = n \ell (k-\ell)_{\ell}^{(n-1)}. \tag{27}$$

*Proof.* The proof follows by taking  $m=1$  in (26).

**Lemma 4.8.** If  $\ell$  is positive real and  $m, n$  and  $t$  are positive integers, then

$$\nabla_{\ell}^m k_{\ell}^{(n)} = \begin{cases} \ell^n n! & \text{if } m = n \\ 0 & \text{if } m > n \end{cases}, \tag{28}$$

*Proof.* The proof follows by induction method on  $m$  and  $n$ .

The following theorem is the generalized version of Newton's formula with reference to  $\nabla_{\ell}$

**Theorem 4.9.** Let  $f(k)$  be an  $n^{\text{th}}$  degree polynomial in  $k$ . Then  $f(k)$  can be expressed as

$$f(k) = f(0) + \frac{\nabla_{\ell} f(0)}{1!\ell} k_{\ell}^{(1)} + \frac{\nabla_{\ell}^2 f(0)}{2!\ell^2} k_{\ell}^{(2)} + \dots + \frac{\nabla_{\ell}^n f(0)}{n!\ell^n} k_{\ell}^{(n)}. \tag{29}$$

*Proof.* Assume that  $f(k) = a_0 + a_1 k_{\ell}^{(1)} + a_2 k_{\ell}^{(2)} + \dots + a_n k_{\ell}^{(n)}$  (30)

The coefficients are determined from the relation

$$\nabla_{\ell}^r f(0) = r! \ell^r a_r. \tag{31}$$

The proof follows by (30) and (31).

**Theorem 4.10.** Let  $f(k)$  be an  $n^{\text{th}}$  degree polynomial in  $k$ . Then  $f(k-t)$  can be expressed as

$$f(k-t) = f(t) + \frac{\nabla_{\ell} f(t)}{1!\ell} (k-t)_{\ell}^{(1)} + \frac{\nabla_{\ell}^2 f(t)}{2!\ell^2} (k-t)_{\ell}^{(2)} + \dots + \frac{\nabla_{\ell}^n f(t)}{n!\ell^n} (k-t)_{\ell}^{(n)}.$$

*Proof.* Replacing  $k$  by  $k-t$  and  $0$  by  $t$  in (31), we get the proof.

#### V. INVERSE OF GENERALIZED BACKWARD DIFFERENCE OPERATER

In this chapter, we define the inverse of generalized backward difference operator and establishes generalized version of Montmorte's Theorem. Also we find the relation between  $k^n$  and  $k_{\ell}^{(n)}$ .

**Definition 5.1.** The inverse of generalized backward difference operator denoted by  $\nabla_{\ell}^{-1}$  is defined as

$$\text{if } \nabla_{\ell} v(k) = u(k), \text{ then } v(k) = \nabla_{\ell}^{-1} u(k) + c_j \tag{32}$$

and the  $n^{\text{th}}$  order inverse operator denoted by  $\nabla_{\ell}^{-n}$  is defined as,

$$\text{if } \nabla_{\ell}^n v(k) = u(k), \text{ then } v(k) = \nabla_{\ell}^{-n} u(k) + c_j, \text{ where } c_j \text{ is constant.}$$

**Lemma 5.2.** If  $\ell$  is a positive real and  $n$  is positive integer, then

$$\nabla_\ell^{-1} k_\ell^{(n)} = \frac{(k+\ell)_\ell^{(n+1)}}{(n+1)\ell} + c_j, \quad j = k - \left\lfloor \frac{k}{\ell} \right\rfloor \ell. \tag{33}$$

*Proof.* The proof follows by Definition 5.1 and the relation

$$\nabla_\ell \left\{ \frac{(k+l)_\ell^{(n+1)}}{(n+1)\ell} + c_j \right\} = k_\ell^{(n)}.$$

**Lemma 5.5.** If  $\ell$  is positive real and  $k \in N_\ell(j)$ , then

$$\nabla_\ell^{-1} v(k) = \sum_{r=0}^{\left\lfloor \frac{k}{\ell} \right\rfloor} v(k-r\ell) + c_j. \tag{34}$$

*Proof.* The proof follows by Definition 5.1 and the relation

$$\nabla_\ell \left\{ \sum_{r=0}^{\left\lfloor \frac{k}{\ell} \right\rfloor} v(k-r\ell) + c_j \right\} = v(k)$$

The following theorem establishes generalized version of Montmort's theorem according to  $\nabla_\ell$ .

**Theorem 5.6.** If the series  $\sum_{k=0}^\infty u(-k\ell) x^{-k\ell}$  converges then it can be expressed as

$$\sum_{k=0}^\infty u(-k\ell) x^{-k\ell} = \sum_{k=0}^\infty \frac{x^{-k\ell} \nabla_\ell^k u(0)}{(1-x^{-\ell})^{k+\ell}} \tag{35}$$

*Proof.* From the shift operator  $E^{-\ell}$ , we get

$$\sum_{k=0}^\infty u(-k\ell) x^{-k\ell} = \sum_{k=0}^\infty x^{-k\ell} E^{-k\ell} u(0) = (1-x^{-\ell} E^{-\ell})^{-\ell} u(0).$$

Now the proof follows from (4).

**Lemma 5.7.** For  $\lambda \neq 1$ ,  $k \geq 2\ell$ , and  $P_k$  is any function of  $k$ , then

$$\sum_{r=0}^{\left\lfloor \frac{k}{\ell} \right\rfloor} \lambda^{k-r\ell} P_{k-r\ell} = \frac{\lambda^k}{(1-\lambda^{-\ell})} \left[ 1 - \frac{\lambda^\ell \nabla_\ell}{1-\lambda^{-\ell}} + \frac{\lambda^{2\ell} \nabla_\ell^2}{(1-\lambda^{-\ell})^2} - \dots \right] P_k.$$

*Proof.* If  $F_k$  is any function of  $k$ , then

$$\nabla_\ell \lambda^k F_k = \lambda^k F_k - \lambda^{k-\ell} F_{k-\ell} = \lambda^k F_k (1 - \lambda^{-\ell} E^{-\ell}) = \lambda^k P_k$$

where  $P_k = (1 - \lambda^{-\ell} E^{-\ell}) F_k$  or  $(1 - \lambda^{-\ell} E^{-\ell})^{-1} P_k = F_k$

Operating  $\nabla_\ell^{-1}$  on both sides of the above relation, we get

$$\nabla_\ell^{-1} (\lambda^k P_k) = \lambda^k F_k + c = \lambda^k (1 - \lambda^{-\ell} E^{-\ell})^{-1} P_k + c.$$

The proof follows by (4) and the Binomial theorem.

**Lemma 5.8.** Relation between  $\nabla_\ell^{-1}$  and  $\nabla^{-1}$  is

$$\sum_{i=0}^{\ell-1} \nabla_\ell^{-1} v(k-i) = \nabla^{-1} v(k) + c . \tag{36}$$

The following is a formula for the finding the sum of the  $n^{\text{th}}$  powers of an Arithmetic Progression.

**Theorem 5.9.** If  $S_r^n$ 's are the Stirling numbers of second kind,  $n$  is a positive integer,  $i$  is any non-negative integer,  $k \in N$  and  $k > n$ , then

$$\sum_{j=0}^{k-1} (j\ell+i)^n = \sum_{r=1}^n S_r^n \frac{\ell^{n-(r+1)}}{(r+1)} \left[ ((k+1)\ell+i)_\ell^{(r+1)} - ((n+1)\ell+i)_\ell^{(r+1)} \right] + \sum_{j=0}^n (j\ell+i)^n . \tag{37}$$

*Proof.* From (26), we can have

$$\nabla_\ell^{-1} k^n = \nabla_\ell^{-1} \left[ \sum_{r=1}^n S_r^n \ell^{n-r} k_\ell^{(r)} \right] + c$$

Applying (33) and (34) on the above relation, we get

$$\sum_{t=0}^{\lfloor \frac{k}{\ell} \rfloor} (k-t\ell)^n = \sum_{r=1}^n S_r^n \ell^{n-r} \frac{(k+\ell)_\ell^{(r+1)}}{(r+1)\ell} + c$$

Replacing  $k$  by  $k\ell+i$

$$\sum_{t=0}^{\lfloor \frac{k\ell+i}{\ell} \rfloor} (k\ell+i-t\ell)^n = \sum_{r=1}^n S_r^n \ell^{n-r-1} \frac{(k\ell+i+\ell)_\ell^{(r+1)}}{(r+1)} + c$$

which reduces to

$$\sum_{j=0}^k (j\ell+i)^n = \sum_{r=1}^n S_r^n \ell^{n-r-1} \frac{(k\ell+i+\ell)_\ell^{(r+1)}}{(r+1)} + c . \tag{38}$$

Put  $k = n$  in (38), we can get

$$c = \sum_{j=0}^k (j\ell+i)^n - \sum_{r=1}^n S_r^n \ell^{n-(r+1)} \frac{((n+1)\ell+i)_\ell^{(r+1)}}{(r+1)} .$$

Substitute the value of  $c$  in (38) we get the required result.

**Remark 5.10.** Equation (37) can be expressed as

$$\sum_{j=n+1}^k (j\ell+i)^n = \sum_{r=1}^n \left\{ \frac{S_r^n \ell^{n-r-1}}{(r+1)} \left[ ((k+1)\ell+i)_\ell^{(r+1)} - ((n+1)\ell+i)_\ell^{(r+1)} \right] \right\} . \tag{39}$$

The following example is the sum of fifth powers of an arithmetic progression.

**Example 5.11.** Find the value of  $20^5 + 23^5 + 26^5 + \dots + 47^5$

**Solution:** In (37), by taking  $\ell = 3$ ,  $i = 2$ ,  $k\ell+i = 47$ ,  $n = 5$ , we get

$$\begin{aligned}
\text{Sum} &= \sum_{r=1}^5 \frac{S_r^5 3^{4-r}}{(r+1)} \left[ (50)_3^{(r+1)} - (20)_3^{(r+1)} \right] \\
&= \frac{S_1^5 3^3}{2} \left[ (50)_3^{(2)} - (20)_3^{(2)} \right] + \frac{S_2^5 3^2}{3} \left[ (50)_3^{(3)} - (20)_3^{(3)} \right] + \\
&\quad \frac{S_3^5 3}{4} \left[ (50)_3^{(4)} - (20)_3^{(4)} \right] + \frac{S_4^5}{5} \left[ (50)_3^{(5)} - (20)_3^{(5)} \right] + \\
&\quad \frac{S_5^5}{(6)3} \left[ (50)_3^{(6)} - (20)_3^{(6)} \right] = 717457775 .
\end{aligned}$$

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