Common Fixed Point and Quasi Contraction Ingeneralised Metric Space of Finite Order υ

Shishir jain
Department of Applied Mathematics
Shri Vaishnav Institute of Technology and Science, Indore, M.P. India

SHOBHA JAIN
Indore, M.P. India

Abstract- In [1], A. Branciari defines a Generalised metric space of finite order υ and establishes Banach contraction principle in it. The object of this paper is to extend this result to Quasi contraction and prove some fixed point results for two weakly compatible self maps satisfying a generalized contractive condition in this space without assuming it to be Hausdorff . Our result generalizes the said result of [1]. All the results presented in this paper are new.

Keywords –Cone metric space, cone rectangular metric space, common fixedpoint, coincidence point and weakly compatible maps.

I. Introduction

[1] A. Branciari a class of generalized metric space of finite order has been developed which contains the class of metric space as a proper subset of it and established Banach contraction principle in the setting of this space. In [2], Kirk and Shahzad proved Caristi's theorem for generalized metric space of order 2. In [10], Lahiri and Das, a quasi contraction result has been established for one self mapping in this space assuming it to be Hausdorff. Papers [3] to [9] represent some related results in a complete cone metric space.

II. Preliminaries

Definition 2.1 [1]: Let X be a nonempty set, υ be a natural number and suppose the mapping $d: X \times X \to R^+$ satisfies:

(a) $0 \leq d(x,y)$, for all $x, y \in X$ and $d(x,y) = 0$, if and only if $x = y$;

(b) $d(x,y) = d(y,x)$, for all $x, y \in X$;

(c) $d(x,y) \leq d(x,x_1) + d(x_1,x_2) + \cdots + d(x_{\upsilon-1},x_\upsilon) + d(x_\upsilon,y)$,

for all distinct $x_1, x_2, \cdots x_\upsilon \in X - \{x, y\}$ and for all $x, y \in X$.

Then (X, d) is called a generalized metric space of order υ. (shortly a υ g.m.s.). if $\upsilon = 2$ it is called a generalized metric space of order 2 and in short a 2-gms.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $\epsilon > 0$ there is a positive integer N_ϵ such that for all $n > N_\epsilon, d(x_n, x) < \epsilon$, then the sequence $\{x_n\}$ is said toconverge to x, and x is called limit of $\{x_n\}$. We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, as $n \to \infty$.

If for every $\epsilon > 0$ there is a positive integer N_ϵ such that for all $n, m > N_\epsilon$, $d(x_n, x_m) < \epsilon$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X. If every Cauchy sequence in X is convergent in X then X is called a complete generalized metric space.

Definition 2.2. [9]: Let A and S be self maps of a set. If $w = Ax = Sx$, forsome $x \in X$, then w is called a point of coincidence A and S.

Definition 2.3. [9]: Let X be any set. A pair of self maps in X is said to be weakly compatible if $u \in X$, $Au = Su$ imply $SAu = ASu$.

Proposition 2.4.: Let (f, g) be a pair of weakly compatible self maps in a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g in X .

III. MAIN RESULTS

Lemma 3.1. Let (X, d) be a generalized metric space of order υ. Let A and S be self mappings on X satisfying:
(3.1.1) $A(X) \subseteq S(X)$;
(3.1.2) For some $\lambda, \mu, \beta \in [0,1)$ with $\lambda + \mu + \beta < 1$,
$d(Ax, Ay) \leq \lambda d(Ax, Sx) + \mu d(Ay, Sy) + \beta d(Ax, Sy)$; *for all x, y in X*.
For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_n = Sx_{n+1} = y_n$; for all n and writing $d_n = d(y_n, y_{n+1})$, for all n > 0, Then
(I) If the pair (A; S) has a point of coincidence then it is unique.
(II) (a) If $y_n = y_{n+1}$; for some n then $Sx_n = Ax_{n+1} = u$ (say) and the pair (A; S) has a unique point coincidence u in X.
(b) If $\lambda = 0$ then $y_n = y_{n+1}$ and the pair (A; S) has a unique point of coincidence u in X.
(III) (a) If $\lambda \neq 0$ then $d_n \leq hd_{n-1}$; for some h in (0, 1), thus $d_n \leq h^n d_0, \forall n$ and if $m \geq 1$ then $d_{n+m} < d_n$.
(b) $y_n = y_{n+p}$ for some p ≥ 1 then p = 1. Thus, if two terms of $\{y_n\}$ are equal, then they are consecutive.

Proof. (of I) Suppose u and v are two points of coincidence the maps A and S. i.e. $Au = Su$ and $Av = Sv$, for some $u, v \in X$. Taking $x = u, y = v$ in (3.1.2), we have
$d(Au, Av) \leq \lambda d(Au, Su) + \mu d(Av, Sv) + \beta d(Au, Sv)$.
i. e., $d(Su, Sv) \leq \beta d(Su, Sv)$.
As $\beta < 1, d(Su, Sv) = 0$. Hence, $Su = Sv$. This proves the uniqueness of point of coincidence of the pair (A, S).

Proof. (of II (a)) By definition of y_n and y_{n+1} and from (I), the result follows.

Proof. (of II (b))Taking $\lambda = 0, x = x_n, y = x_{n+1}$ in (3.1.2) we get,
$d(Ax_n, Ax_{n+1}) \leq \mu d(Ax_{n+1}, Sx_{n+1}) + \beta d(Ax_n, Sx_{n+1})$
i.e. $d(y_n, y_{n+1}) \leq \mu d(y_{n+1}, y_n) + \beta d(y_n, y_n) = \mu d(y_{n+1}, y_n)$
Thus, $(1 - \mu)d_n \leq 0$. As $\mu < 1$ we get, $d_n = 0$. Hence $y_n = y_{n+1}$ and the pair (A, S) has a unique point of coincidence u in X.

Proof. (of III (a)) : Taking $x = x_n, y = x_{n+1}$ in (3.1.2) we get,
$d(Ax_n, Ax_{n+1}) \leq \lambda d(Ax_n, Sx_n) + \mu d(Ax_{n+1}, Sx_{n+1}) + \beta d(Ax_n, Sx_{n+1})$
i.e.,

$$\begin{aligned} d(y_n, y_{n+1}) &\leq \lambda d(y_n, y_{n-1}) + \mu d(y_{n+1}, y_n) + \beta d(y\_n, y\_n) \\ &= \lambda d(y_n, y_{n-1}) + \mu d(y_{n+1}, y_n) \end{aligned}$$

Hence,
$d_n \leq \lambda d_{n-1} + \mu d_n$,
i.e., $(1 - \mu)d_n \leq \lambda d_{n-1}$. Thus
(3.1) $$d_n \leq h d_{n-1},$$

Where $h = \frac{\lambda}{1-\mu}$, in view of (3.1.2), $0 < h < 1$ *as* $\lambda < 1$.

Proof. (of III (b)) Suppose $y_n = y_{n+p}$ for some $p > 1$ Then,
$d(y_n, y_{n+1}) = d(y_{n+p}, y_{n+1}) = d(Ax_{n+p}, Ax_{n+1})$
Taking $x = x_{n+p}, y = x_{n+1}$ in (3.1.2) we get,

$$d(Ax_{n+p}, Ax_{n+1}) \leq \lambda d(Ax_{n+p}, Sx_{n+p}) + \mu d(Ax_{n+1}, Sx_{n+1}) + \beta d(Ax_{n+p}, Sx_{n+1})$$

Thus,
$d_n \leq \lambda d_{n+p-1} + \mu d_n + \beta d(y_{n+p}, y_n)$,
$= \lambda d_{n+p-1} + \mu d_n$.
Hence, $d_n \leq \frac{\lambda}{1-\mu} d_{n+p-1} = h\, d_{n+p-1}$. This gives $d_n < d_{n+p-1}$ and $p - 1 \geq 1$. Thus $d_n < d_{n+m}$, for some $m \geq 1$. This contradicts III(a). Hence $p = 1$ and our result follows

Theorem 3.2. Let (X,d) be a generalized metric space of order υ. Let A and S be self-mappings on X satisfying (3.1.1),(3.1.2) and
(3.2.1). Either $A(X)$ or $S(X)$ is complete;
Then A and S have a unique point of coincidence in X. Moreover, if the pair (A,S) is weakly compatible, then this coincidence point is the unique common fixed point of the maps A and S in X.

Proof. Construct sequences $\{x_n\}$ and $\{y_n\}$ in X as defined in Lemma 3.1. If $y_n = y_{n+1}$, for some n, then from then from Lemma 3.1 the maps A and S have a uniquepoint of coincidence in X and that the elements of the sequence $\{y_n\}$ are all distinctor else some consecutive terms are equal . Now, it remains to prove the existence of unique point of coincidence when the terms of $\{y_n\}$ are all distinct. First we showthat $\{y_n\}$ is a Cauchy sequence in X.To see this we apply induction on m to show that for all positive integers m and n.
(3.2) $\qquad d(y_n, y_{n+m}) < h^{n-1}d_0$.
From Lemma 3.1 III (a) and equation (3.2) is true for $m = 1$. Suppose (induction hypothesis) that
(3.2) $\qquad d(y_n, y_{n+m-1}) < h^{n-1}d_0$.
Taking $x = x_n,\ y = x_{n+m}$ in (3.1.2)

$$d(Ax_n, Ax_{n+m}) \le \lambda\, d(Ax_n, Sx_n) + \mu\, d(Ax_{n+m}, Sx_{n+m}) + \beta\, d(Ax_n, Sx_{n+m}),$$

i.e.,

$$\begin{aligned} d(y_n, y_{n+m}) &\le \lambda d(y_n, y_{n-1}) + \mu d(y_{n+m}, y_{n+m-1}) + \beta\, d(y_n, y_{n+m-1}) \\ &= \lambda d_{n-1} + \mu\, d_{n+m-1} + \beta d(y_n, y_{n+m-1}) \\ &< \lambda h^{n-1}d_0 + \mu h^{n+m-1}d_0 + \beta h^{n-1}d_0, \\ &< (\lambda + \mu + \beta)h^{n-1}d_0, \\ &< h^{n-1}d_0, \end{aligned}$$

as $\lambda + \mu + \beta < 1$
Thus equation (3.2) is true for for all m and n and hence $\{y_n\}$ is a Cauchy sequencein $S(X) \cap A(X)$. Now, we show that the mappings A and S have a unique point of coincidence.
Case I: $S(X)$ is complete
In this case $\{y_n\} = \{Sx_{n+1}\}$ is a Cauchy sequence in $S(X)$, which is complete. So $\{y_n\} \to z \in S(X)$. Hence there exist $u \in X$ such that $z = Su$. As all elements of $\{y_n\}$ are distinct, taking n sufficiently large we get $y_n, y_{n+1}, y_{n+2} \cdots y_{n+v-1} \in X - \{Au, Su\}$
Now,

$$\begin{aligned} d(Au, Su) &\le d(Au, y_n) + d(y_n, y_{n+1}) + d(y_{n+v-1}, Su), \\ &= d(Au, y_n) + d_n + d_{n+1} \cdots d_{n+v-2} + d(y_{n+v-1}, Su), \\ &\le d(Ax_n, Au) + h^n d_0 + h^{n+1}d_0 + \cdots + h^{n+v-2}d_0 + d(y_{n+v-1}, Su), \\ &\le \lambda d(Ax_n, Sx_n) + \mu d(Au, Su) + \beta d(Ax_n, Su) + \frac{h^n d_0}{1-h} + d(y_{n+v-1}, Su). \end{aligned}$$

Thus,

$$\begin{aligned} (1-\mu)d(Au, Su) &\le \lambda d(y_n, y_{n-1}) + \beta d(y_n, Su) + \frac{h^n d_0}{1-h} + d(y_{n+v-1}, Su), \\ &\le \lambda\, h^{n-1}d_0 + \beta\, d(y_n, Su) + \frac{h^n d_0}{1-h} + d(y_{n+v-1}, Su) \end{aligned}$$

As, $y_n, y_{n+v-1} \to Su$, and $h < 1$, we have $(1-\mu)d(Au, Su) = 0$ Thus, $Au = Su$. Thus the pair (A,S) has a point of coincidence $z = Au = Su$.
Case II: $A(X)$ is complete.
In this case $\{y_n\} = \{Ax_n\}$ is a Cauchy sequence in $A(X)$ which is complete. So $\{y_n\} \to z = Aw$, for some $w \in X$. As $A(X) \subseteq S(X)$ there exists $v \in X$ such that $Aw = Sv$. Thus $\{y_n\} \to Sv$. It follows from case I that $Av = Sv$.
Thus in both the cases the pair (A,S) has a point of coincidence, which is unique inview of Lemma 3.1 As (A,S) is weakly compatible from Proposition 2.4, it follows that the point of coincidence of A and S is their unique common fixed point in X.
Theorem 3.3. Let (X,d) be a generalized metric space of order υ. Let A and S be self-mappings on X satisfying (3.1.2), (3.2.1) and
(3.3.1). For some $\lambda, \mu, \alpha \in [0,1)$ with $\lambda + \mu + \alpha < 1$,

$$d(Ax, Ay) \le \lambda\, d(Ax, Sx) + \mu\, d(Ay, Sy) + \alpha\, d(Ay, Sx), \forall x, y \in X.$$

Then A and S have a unique point of coincidence in X. Moreover, if the pai(rA, S) is weakly compatible, then this point of coincidence is the unique common fixed point of the maps A and S in X.

Proof. We have,

$d(Ax, Ay) = d(Ay, Ax) \leq \lambda\, d(Ay, Sy) + \mu\, d(Ax, Sx) + \alpha\, d(Ax, Sy)$.

Interchanging λ and μ and writing α as β, we get

$d(Ax, Ay) \leq \lambda\, d(Ax, Sx) + \mu\, d(Ay, Sy) + \beta\, d(Ax, Sy)$.

Rest follows from Theorem 3.2.

Taking $\lambda = \mu = k$ and $\alpha = 0$ in Theorem 3.2, we have the following result:

Corollary 3.4. Let (X, d) be a generalized metric space of order υ. Let A and S be self mappings on X satisfying (3.1.1) and for some $k \in [0, 1/2)$ and for all $x, y \in X$,

$d(Ax, Ay) \leq k[\, d(Ax, Sx) + d(Ay, Sy)]$.

If $A(X)$ or $S(X)$ is complete and the pair (A, S) is weakly compatible then themappings A and S have a unique common fixed point in X.

Remark 3.5. The above corollary is known to true in cone metric space withrespect to a cone P in a real Banach space with non empty interior as established in [5] by Huang and Zhang.

Theorem 3.6. Let (X, d) be a generalized metric space of order υ. Let A and S be self mappings on X satisfying (3.1.2), (3.2.1) and

(3.6.1) For some $\delta \in (0, 1)$, $d(Ax, Ay) \leq \delta\, d(Sx, Sy)$, $\forall x, y \in X$.

Then A and S have a unique point of coincidence in X. Moreover, if the pair $(A, S)$is weakly compatible, then this coincidence point is the unique common fixed pointof the maps A and S in X.

Proof. For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X as defined in Lemma3.1. From (3.6.1) it is easy to see that$d_n \leq \delta d_{n-1}, \forall n$and thus$d_n \leq \delta^n d_0$. Further,it is easy to see that the axioms (I), II(a), III(b) of Lemma 3.1 holds good. On thelines of the proof of Theorem 2.2 of [1], we can show that

(3.4) $$d(y_n, y_{n+m}) < \frac{\delta^n}{1-\delta} M,$$

where, $M = max\{d(y_0, y_1), d(y_0, y_2), \cdots, d(y_0, y\_v)\}$ It remains to prove the existence of a point of coincidence of A and S when the terms of the sequence $\{y_n\}$ areall distinct. From 3.4 it follows that $\{y_n\}$ is a Cauchy sequence in $A(X) \cap S(X)$. As in Case I and II of Theorem 3.2 we see that mappings A and S have a uniquepoint of coincidence which is unique in view of Lemma 3.1. Further from Proposition 2.4 this point of coincidence is the unique fixed point of A and S if they are weak compatible.

Remark 3.7. Taking $S = I$, the identity mapping in Theorem 3.6 the results of[1], A. Branciari follow.

Keeping one of the constants $\alpha, \beta, \lambda, \mu$ non-zero and all others equal to zero in Theorems 3.2 , 3.3 and using Theorem 3.6 we have the following result:

Theorem 3.8. Let (X, d) be a generalized metric space of order v. Let A and S betwo weakly compatible self-mappings on X satisfying (3.1.1) and (3.2.1). Then $A$and S have a unique common fixed point in X if for a fixed $\lambda \in (0, 1)$ and $\forall x, y \in X$, $d(Ax, Ay) \leq \lambda u$, where $u \in \{d(Ax, Sx), d(Ay, Sy), d(Sx, Sy), d(Ax, Sy)\}$.

The following definition and Theorem appears in D. Ilic, V. Rakocevic [6]

Definition[6](Quasi contraction) A self-map f on a cone metric space (X, d) is said to be a quasi-contraction if for a fixed $\lambda \in (0, 1)$, $d(fx, fy) \leq \lambda u$, $\forall x, y \in X$, where $u \in \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$.

Theorem 2.1 [6]. Let (X, d) be a complete cone metric space and P be normalcone. Then a Quasi contraction f has a unique fixed point in X for each $x \in X$, the iterative sequence $\{f_n(x)\}$ converges to this fixed point.

Remark 3.9. Taking $S = I$, the identity mapping in Theorem 3.8 it follows that the above result of [6] is true for generalized metric space of order v . The same result has been established in [3] for a complete metric space. In Lahiri and Das[10], this quasi contraction result has been established for one self-mapping in a generalized metric space of order 2 assuming the space to be Hausdorf.

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