Turán - Type Inequalities for the Central Moments of Some Continuous Probability Distributions via the Cauchy-Bunyakovsky-Schwarz Inequality

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Abstract - We establish new inequalities for the central moments of some continuous probability distributions, using a form of the Cauchy-Bunyakovsky-Schwarz inequality. Our new inequalities extend the class of the Turán-type inequalities.

Keywords: Cauchy-Bunyakovsky-Schwarz inequality, Turán-type inequalities, Central moments, Weibull distribution, Exponential distribution, Gaussian distribution, Erlang Distribution.

1. INTRODUCTION

The well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality states that

$$\left(\int_a^b u(t)\,dt\right)\left(\int_a^b v\,(t)\,dt\right) \geq \left(\int_a^b u^{\frac{1}{2}}\,(t)\,v^{\frac{1}{2}}\,(t)dt\right)^2, \quad (1.1)$$

for every function $u, v\ : [a,b] \to [0,\infty)$, such that the integrals exist.

A. Laforgia and P. Natalini [1] used the following form of the CBS inequality (1.1):

$$\left(\int_a^b g(t)f^m(t)\,dt\right)\left(\int_a^b g(t)\,f^n(t)\,dt\right) \geq \left(\int_a^b g\,(t)\,f^{\frac{m+n}{2}}(t)dt\right)^2 \quad (1.2)$$

to establish some new Turán-type inequalities involving the special functions such as Gamma, or Polygamma functions and Riemann zeta function.

Here, g, f: $[a, b] \to [0, \infty)$ are such that the involved integrals in (1.2) exist.

The importance, in many fields of mathematics, of the inequalities of the type $\boldsymbol{f_n(x)f_{n+2}(x) - f^2{}_{n+1}(x) \geq 0}$, where $n = 0,1,2, \ldots.,$ is well known. They are named, by Karlin and Szeg$\ddot{o}$, Turán-type inequalities because the first of this type of inequalities was proved by Turán [2].

In this paper, we prove Turán-type inequalities for the central moments of some continuous probability distributions such as Exponential distribution or negative Exponential distribution, Normal distribution, Erlang Distribution and Weibull distribution, using the following form of the CBS inequality (1.2):

$$\left(\int_a^b f(x)g^{2m}(x)\,dx\right)\left(\int_a^b f(x)\,g^{2n}(x)\,dx\right) \geq \left(\int_a^b f\,(x)\,g^{m+n}(x)dx\right)^2 \quad (1.3)$$

Here $f\ and\ g$ are two non-negative functions of a real variable and $m\ and\ n$ belong to a set S of real numbers, such that the involved integrals in (1.3) exist.

The r^{th} central moment (Moment about mean) of a continuous probability distribution is defined by

$$\boldsymbol{\mu_r} = \int_{-\infty}^{\infty}(x-\bar{x})^r\,f_X(x)\,dx$$

where $\bar{x}$ is the mean and $f_X(x)$ is probability density function of the distribution.

In the next section, we show how to use the inequality (1.3) to establish Turán-type inequalities for the central moments of some continuous probability distributions.

2. THE RESULTS

Definition1. A continuous random variable X is said to follow an Exponential distribution or negative Exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

The r^th central moment of the Exponential distribution is given by

$$\boldsymbol{\mu}_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f_X(x)\,dx = \int_0^{\infty} (x-\bar{x})^r \lambda e^{-\lambda x}\,dx$$

Now if $f(x) = \lambda e^{-\lambda x}$ and $g(x) = (x-\bar{x})$ are substituted in inequality (1.3) for $[a,b] \rightarrow [0,\infty)$, the following inequality is derived

$$\left(\int_0^{\infty} (x-\bar{x})^{2m} \lambda e^{-\lambda x}\,dx\right)\left(\int_0^{\infty} (x-\bar{x})^{2n} \lambda e^{-\lambda x} dx\right) \geq \left(\int_0^{\infty} (x-\bar{x})^{m+n}\ \lambda e^{-\lambda x} dx\right)^2$$

$$\Rightarrow (\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

This can be presented as the following:

Theorem 2.1 For every positive integer $m, n \geq 1$, and for every real number $x \in [0,\infty)$, it holds the Turán-type inequality for the central moments of the Exponential distribution or negative Exponential distribution with parameter $\lambda > 0$:

$$(\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

Remark1. When $m = n + 1$, we find

$$\frac{\boldsymbol{\mu}_{2n}}{\boldsymbol{\mu}_{2n+1}} \geq \frac{\boldsymbol{\mu}_{2n+1}}{\boldsymbol{\mu}_{2n+2}}\ , n = 1,2,3 \ldots \ldots \ldots$$

Definition2. A continuous random variable X is said to have a Normal distribution or Gaussian distribution with parameters μ (called 'mean') and σ^2(called 'variance'), if its probability density function is given by the probability law:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \qquad (-\infty < x < \infty)$$

The r^th central moment of the normal distribution is given by

$$\boldsymbol{\mu}_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f_X(x)\,dx = \int_{-\infty}^{\infty} (x-\mu)^r \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx,$$

Then

$$\boldsymbol{\mu}_{2r} = \int_{-\infty}^{\infty} (x-\mu)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 2\int_0^{\infty} (x-\mu)^{2r} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Hence, if $f(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $g(x) = (x-\mu)$ and $[a,b] \rightarrow [0,\infty)$, are considered in inequality (1.3) then we get

$$\left(\int_0^{\infty} 2(x-\mu)^{2m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right)\left(\int_0^{\infty} 2(x-\mu)^{2n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right) \geq \left(\int_0^{\infty} 2(x-\mu)^{m+n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx\right)^2$$

$$\Rightarrow (\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

This can be presented as the following:

Theorem 2.2 For every positive integer $m, n \geq 1$ such that $m + n$ is even, and for every real number $x \in [0,\infty)$, It holds the Turán-type inequality for the central moments of the Normal distribution or Gaussian distribution with parameters μ and σ^2:

$$(\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

Definition3. A continuous random variable X is said to follow an Erlang Distribution or general Gamma distribution with parameter $\lambda > 0, k > 0$; if its probability density function is given by

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}, & x \geq 0 \end{cases}$$

The r[th] central moment of the Erlang Distribution is given by

$$\boldsymbol{\mu}_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f_X(x)\,dx = \int_0^{\infty} (x-\bar{x})^r \frac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}\,dx$$

By taking $f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}$, $g(x) = (x-\bar{x})$ and $[a,b] \rightarrow [0,\infty)$ in inequality (1.3), we obtain

$$\left(\int_0^{\infty} (x-\bar{x})^{2m} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}\,dx\right)\left(\int_0^{\infty} (x-\bar{x})^{2n} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}\,dx\right) \geq \left(\int_0^{\infty} (x-\bar{x})^{m+n} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{k!}\,dx\right)^2$$

$$\Rightarrow (\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

This can be presented as the following:

Theorem 2.3 For every positive integer $m, n \geq 1$, and for every real number $x \in [0,\infty)$, it holds the Turán-type inequality for the central moments of the Erlang Distribution or general Gamma distribution with parameter $\lambda > 0, k > 0$:

$$(\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

Definition4. A continuous random variable X is said to have a Weibull distribution with parameters $\alpha > 0$ & $\beta > 0$, if its probability density function is given by

$$f_X(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}};\ x > 0$$

The r[th] central moment of the Weibull distribution is given by

$$\boldsymbol{\mu}_r = \int_{-\infty}^{\infty} (x-\bar{x})^r f_X(x)\,dx = \int_0^{\infty} (x-\bar{x})^r \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\,dx$$

Now if $f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}$ and $g(x) = (x-\bar{x})$ are substituted in inequality (1.3) for $[a,b] \rightarrow (0,\infty)$, the following inequality is derived

$$\left(\int_0^{\infty} (x-\bar{x})^{2m} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\,dx\right)\left(\int_0^{\infty} (x-\bar{x})^{2n} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\,dx\right) \geq \left(\int_0^{\infty} (x-\bar{x})^{m+n} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}\,dx\right)^2$$

$$\Rightarrow (\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

This can be presented as the following:

Theorem 2.4 For every positive integer $m, n \geq 1$, and for every real number $x \in (0,\infty)$, it holds the Turán-type inequality for the central moments of the Weibull distribution with parameters $\alpha, \beta > 0$:

$$(\boldsymbol{\mu}_{2m})\,(\boldsymbol{\mu}_{2n}) \geq (\boldsymbol{\mu}_{m+n})^2$$

3. CONCLUDING REMARK

Similarly Turán-type inequalities can also be obtained for the moments about origin and moments about any point ‘a’ by replacing $\bar{x}$ (mean) by 0 and‘a’ (any point) respectively.

4. REFERENCES

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