Existence Results for Nonlinear Boundary Value Problems of Impulsive Fractional Integrodifferential Equations

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Abstract - In this paper, we investigate the existence result for nonlinear impulsive fractional integro-differential equations with boundary conditions by using fixed point theorem and Green's function.

Keywords: Boundary value problem; impulsive; fractional differential equation; Green's function; existence; fixed point theorem.

I. INTRODUCTION

The topic of fractional differential equations has received a great deal of attention from many scientists and researchers during the past decades; see [1-7]. This is mostly due to the fact that fractional calculus provides an efficient and excellent instrument to describe many practical dynamical phenomena which arise in engineering and science such as physics, chemistry, biology, economy, viscoelasticity, electrochemistry, electromagnetic, control, porous media; see [8-13]. Moreover, many researchers study the existence of solutions for fractional differential equations; see [14-16] and the references therein.

In particular, several authors have considered a nonlocal Cauchy problem for abstract evolution differential equations having fractional order. Indeed, the nonlocal Cauchy problem for abstract evolution differential equations was studied by Byszewski [17, 18] initially. Afterwards, many authors [19-21] discussed the problem for different kinds of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces. Balachandran et al.[22, 23] established the existence of solutions of quasilinear integrodifferential equations with nonlocal conditions. N'Guerekata [24] and Balachandran and Park [25] researched the existence of solutions of fractional abstract differential equations with a nonlocal initial condition. Ahmad [26] obtained some existence results for boundary value problems of fractional semi linear evolution equations. Recently, Balachandran and Trujillo [27] have investigated the nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces.

On the other hand, the theory of impulsive differential equations for integer order has emerged in mathematical modeling of phenomena and practical situations in both physical and social sciences in recent years. One can see a significant development in impulsive theory. We refer the readers to [28-31] for the general theory and applications of impulsive differential equations. Besides, some researchers [32-36] have addressed the theory of boundary value problems for impulsive fractional differential equations.

Motivated by the aforementioned works, in this paper, we deal with the existence of solutions of nonlinear boundary value problem of fractional impulsive integro-differential equations:

$$\begin{cases} {}^cD_{0_+}^q x(t) = \omega(t) f\big(t, x(t), x'(t)\big) + \int_0^t g\big(t, s, x(s)\big) ds, \quad 1 < q \le 2, t \in J_1 = J \setminus \{t_1, t_2, \dots, t_n\}, \\ \Delta x|_{t=t_k} = I_k\big(x(t_k)\big),\ \Delta x'|_{t=t_k} = J_k\big(x(t_k)\big),\ t_k \in (0,1),\ k = 1,2,\dots,n, \quad \dots\dots \quad (1.1) \\ \alpha_1 x(0) - \beta_1 x'(0) = 0,\ \alpha_2 x(1) + \beta_2 x'(1) = 0. \end{cases}$$

where ${}^cD_{0_+}^q$ is the Caputo fractional derivative, $J = [0,1]$, $\omega(t) : J \to R_+$ is a continuous function, $f: J \times R \times R \to R$ is a continuous function, $g: J \times J \times R \to R$ is a continuous function, $I_k, J_k: R \to R$ are continuous functions, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$ and $\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \neq 0$. $\Delta x|_{t=t_k} = x(t)_k^+ - x(t)_k^-$ with $x(t_k)^+ = \lim_{h\to 0^+}\big(x(t_k + h)\big)$, $x(t_k)^- = \lim_{h\to 0^-}\big(x(t_k + h)\big)$, $k = 1,2,3,\dots,n$, $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = 1$. $\Delta x'|_{t=t_k}$ has a similar meaning for $x'(t)$.

The rest of this paper is organized as follows. In section 2, we present some notations and preliminary results about fractional calculus and differential equations to be used in the following section. In section 3, we present the expression and properties of Green's function associated with problem (1.1). In section 4, we get some existence results for problem (1.1) by means of standard fixed point theorem. Finaly we shall give an illustrative example for our results.

II. PRELIMINARIES

In this section, we give some preliminaries for discussing the solvability of problem(1.1).

Definition 2.1. The fractional (arbitrary) order integral of the function $h \in L^1(J, R_+)$ of order $q \in R_+$ is defined by

$$I_{0^+}^q h(t) = \frac{1}{\Gamma(q)} \int_0^1 (t-s)^{q-1} h(s) ds,$$

where $\Gamma(.)$ is the Euler Gamma Function.

Definition 2.2. For a function h given on the interval J, the Caputo-type fractional derivative of order $q > 0$ is defined by

$${}^cD_{0^+}^q h(t) = \frac{1}{\Gamma(n-q)} \int_0^1 (t-s)^{n-q-1} h^n(s) ds,\ n = [q] + 1,$$

where the function $h(t)$ has absolutely continuous derivatives up to order $(n-1)$.

Lemma 2.1 Let > 0, then the differential equation

$${}^cD^q h(t) = 0$$

has the following solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \dots + c_{(n-1)} t^{n-1}, \qquad c_i \in R, i = 0,1,2,\dots,n-1, n = [q] + 1.$$

Lemma 2.2 Let > 0, then

$$I^q\, {}^cD_{0^+}^q h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{(n-1)} t^{n-1},$$

for some $c_i \in R,\ i = 0,1,2,\dots,n-1, n = [q] + 1.$

We define the set of functions as follows

Let $J' = [0,1] \setminus \{t_1, t_2, \dots\dots, t_n\}$ and

$$PC[J, R] = \{x : J \to R;\ x \in C\big((t_k, t_{k+1}), R\big), x(t_k^+) \text{ and } x(t_k^-) \text{ exist with } x(t_k^-) = x(t_k),\ k = 1,2,3,\dots,n\}$$

$PC^1[J,R] = \{x' \in PC[J,R];\ x'(t_k^{\pm})$ *and* $x'(t_k^{-})$ *exist and* x' *is left continuous at* t_k,
$k = 1,2,3,\dots,n\}$

Then $PC[J,R]$ is a Banach space with the norm

$$\|x\|_{PC} = \sup_{t\in J}|x(t)|,$$

$PC^1[J,R]$ is a Banach space with the norm

$$\|x\|_{PC^1} = \max\{\|x\|_{PC},\ \|x'\|_{PC}\}.$$

III. EXPRESSION AND PROPERTIES OF GREEN'S FUNCTION

Consider the following fractional impulsive boundary value problem :

$$\begin{cases} {}^cD_{0_+}^q x(t) = \sigma(t), 1 < q \le 2,\ t \in J_1 = J\setminus\{t_1, t_2, \dots, t_n\}, \\ \Delta x|_{t=t_k} = I_k(x(t_k)),\ \Delta x'|_{t=t_k} = J_k(x(t_k)),\ t_k \in (0,1),\ k = 1,2,\dots,n, \\ \alpha_1 x(0) - \beta_1 x'(0) = 0,\ \alpha_2 x(1) + \beta_2 x'(1) = 0. \end{cases} \quad \dots\dots\dots\dots(3.1)$$

Proposition 3.1 [36] The solution of problem (3.1) can be expressed by

$$\begin{aligned} x(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \sum_{i=1}^{n+1} G'_{1s}(t.t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\sigma(s)\sigma ds \\ &- \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds + \sum_{i=1}^{n} G'_{1s}(t,t_i)I_i(x(t_i)) \\ &- \sum_{i=1}^{n} G_1(t,t_i)J_i(x(t_i)), t\epsilon\,(t_k,t_{k+1}), \dots\dots\dots \qquad (3.2) \end{aligned}$$

$k = 0,1,2,\dots n,\ \ t_0 = 0, t_{n+1} = 1$.

where

$$G_1(t,s) = -\frac{1}{\eta}\begin{cases}(\beta_1 + \alpha_1 t)(\alpha_2 + \beta_2 - \alpha_2 s), & t \le s, \\ (\beta_1 + \alpha_1 s)(\alpha_2 + \beta_2 - \alpha_2 t), & s \le t,\end{cases} \quad \dots\dots\dots\dots\dots(3.3)$$

and

$$\eta = \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 \quad \dots\dots\dots\dots \quad (3.4)$$

Proof:

suppose that x is a solution of (3.1). Then, for some constants $b_0,\ b_1 \in R$, we have

$$x(t) = I_{0^+}^q \sigma(t) - b_0 - b_1 t$$

$$= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_0 - b_1 t,\ t \in [0,1]. \ \dots\dots\dots\dots\dots(3.5)$$

$$x'(t) = \int_0^t \frac{(t-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds - b_1. \quad \dots\dots\dots \quad (3.6)$$

If $t \in (t_1, t_2]$, then for some constants $c_0,\ c_1 \in R$ we can write

$$x(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_0 - c_1(t-t_1),$$

$$x'(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - c_1.$$

Using impulse conditions $\Delta x|_{t=t_1} = I_1(x(t_1)),\ \Delta x'|_{t=t_1} = J_1(x(t_1)),$

$$-c_0 = \int_0^{t_1} \frac{(t_1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds - b_0 - b_1 + I_1(x(t_1)),$$

$$-c_1 = \int_0^{t_1} \frac{(t_1-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds - b_1 + J_1(x(t_1)).$$

Thus

$$x(t) = \int_{t_1}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \int_0^{t_1} \left(\frac{(t_1-s)^{q-2}(t-t_1)}{\Gamma(q-1)} + \frac{(t_1-s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds - b_o - b_1 t + J_1(x(t_1))(t-t_1) + I_1(x(t_1))$$

$$x'(t) = \int_{t_1}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \int_0^{t_1} \left(\frac{(t_1-s)^{q-2}}{\Gamma(q-1)} \right) \sigma(s)ds - b_1 + J_1(x(t_1)).$$

If $\in (t_k, t_{k+1}]$, repeating the above procedure, we obtain

$$x(t) = \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i-s)^{q-2}(t-t_i)}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds - b_0 - b_1 t + \sum_{i=1}^{k} J_i(x(t_i))(t-t_i) + \sum_{i=1}^{k} I_i(x(t_i)) \ldots\ldots\ldots\ldots\ldots\ldots \quad (3.7)$$

It follows that $x(0) = -b_0,\ x'(0) = -b_1$ and

$$x(1) = \int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i-s)^{q-2}(1-t_i)}{\Gamma(q-1)} + \frac{(t_i-s)^{q-1}}{\Gamma(q)} \right) \sigma(s)ds - b_0 - b_1 + \sum_{i=1}^{k} J_i(x(t_i))(1-t_i) + \sum_{i=1}^{k} I_i(x(t_i)),$$

$$x'(1) = \int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \right) \sigma(s)ds - b_1 + \sum_{i=1}^{k} J_i(x(t_i)).$$

By the boundary conditions, we have

$$b_0 = \frac{1}{\eta}\left\{ \sigma_2\beta_1 \left[\int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i - s)^{q-2}(1-t_i)}{\Gamma(q-1)} + \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right)\sigma(s)ds \right.\right.$$
$$\left. + \sum_{i=1}^{k} J_i(x(t_i))(1-t_i) + \sum_{i=1}^{k} I_i(x(t_i)) \right]$$
$$\left. + \beta_1\beta_2 \left[\int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right)\sigma(s)ds + \sum_{i=1}^{k} J_i(x(t_i)) \right] \right\} \quad (3.8)$$

$$b_1 = \frac{1}{\eta}\left\{ \alpha_1\alpha_2 \left[\int_{t_n}^{1} \frac{(1-s)^{q-1}}{\Gamma(q)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i - s)^{q-2}(1-t_i)}{\Gamma(q-1)} + \frac{(t_i - s)^{q-1}}{\Gamma(q)} \right)\sigma(s)ds \right.\right.$$
$$\left. + \sum_{i=1}^{k} J_i(x(t_i))(1-t_i) + \sum_{i=1}^{k} I_i(x(t_i)) \right]$$
$$\left. + \alpha_1\beta_2 \left[\int_{t_n}^{1} \frac{(1-s)^{q-2}}{\Gamma(q-1)} \sigma(s)ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \right)\sigma(s)ds + \sum_{i=1}^{k} J_i(x(t_i)) \right] \right\} \quad (3.9)$$

where η is defined in (3.4).

Substituting (3.8), (3.9) into (3.7), we obtain (3.2). This completes the proof.

Proposition 3.2 For all $t, s \in J$, we have

$$|G_1(t,s)| \le \frac{2(|\beta_1\alpha_2| + |\alpha_1\alpha_2|) + |\beta_1\beta_2| + |\alpha_1\beta_2|}{|\eta|} \ldots\ldots\ldots \qquad 3.10$$

Proposition 3.3 For all $t, s \in J$, we have

$$\begin{cases} |G'_{1s}(t,s)| \le max\left\{\dfrac{|\beta_1\alpha_2| + |\alpha_1\alpha_2|}{|\eta|}, \dfrac{|\alpha_1\beta_2| + 2|\alpha_1\alpha_2|}{|\eta|}\right\}, \\ \qquad\qquad J \\ |G'_{1t}(t,s)| \le max\left\{\dfrac{|\beta_2\alpha_1| + 2|\alpha_1\alpha_2|}{|\eta|}, \dfrac{|\alpha_2\beta_1| + |\alpha_1\alpha_2|}{|\eta|}\right\}, \ldots\ldots\ldots \\ \quad \text{and} \ \ |G''_{1st}(t,s)| = \dfrac{|\alpha_1\alpha_2|}{|\eta|}. \end{cases} \qquad 3.11$$

for the sake of convenience, let

$$c_1 = \frac{2(|\beta_1\alpha_2| + |\alpha_1\alpha_2|) + |\beta_1\beta_2| + |\alpha_1\beta_2|}{|\eta|}, \ldots\ldots\ldots \qquad (3.12)$$

$$c_2 = max\left\{\frac{|\beta_1\alpha_2| + |\alpha_1\alpha_2|}{|\eta|}, \frac{|\alpha_1\beta_2| + 2|\alpha_1\alpha_2|}{|\eta|}\right\}, c_3 = \frac{|\alpha_1\alpha_2|}{|\eta|} \dots\dots \quad (3.13)$$

Then it follows from (3.5)-(3.8) that

$$|G_1(t,s)| \le c_1, \left|G'_{1s}(t,s)\right| \le c_2, \left|G'_{1t}(t,s)\right| \le c_2, \left|G''_{1st}(t,s)\right| \le c_3 \quad \dots \quad (3.14)$$

Definition 3.1 A functions $x \epsilon PC^1[J,R] \cap C^2[J',R]$ with its caputo derivative of order q existing on J is a solution of problem (1.1) if it satisfies (1.1).

We give the following hypotheses:

(H_1) $\omega : J \to [0,+\infty)$ is a continuous function and there exits $t_0 \in J$ such that $\omega(t_0) > 0$;

(H_2) $g : J \times J \times R \to R$ is continuous function;

(H_3) $f : J \times R \times R \to R$ is continuous function;

(H_4) $I_k, J_k : R \to R$ are continuous functions.

It follows from Proposition (3.1) that

Lemma 3.1 If $(H_1) - (H_4)$ hold, then a function $x \epsilon PC^1[J,R] \cap C^2[J',R]$ is a solution of problem (1.1) iff $x \in PC^1[J,R]$ is a solution of the impulsive fractional integral equation.

$$\begin{aligned} x(t) = &\int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\left[\omega(s)f(s,x(s),x'(s)) + \int_0^s g(s,\sigma,x(\sigma))d\sigma\right]ds \\ &+ \sum_{i=1}^{n+1} G_{1s}^{i}(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\left[\omega(s)f(s,x(s),x'(s)) + \int_0^s g(s,\sigma,x(\sigma))d\sigma\right]ds \\ &- \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left[\omega(s)f(s,x(s),x'(s)) + \int_0^s g(s,\sigma,x(\sigma))d\sigma\right]ds \\ &+ \sum_{i=1}^{n} G_{1s}^{i}(t,t_i)I_i(x(t_i)) - \sum_{i=1}^{n} G_1(t,t_i)J_i(x(t_i)), \quad t \epsilon (t_k,t_{k+1}), k = 0,1,2,\dots n, \\ & t_0 = 0, t_{n+1} = 1 \dots\dots\dots\dots (3.15) \end{aligned}$$

Define $T : PC^1[J,R] \to PC^1[J,R]$ *by*

$$(Tx)(t) = \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left[\omega(s) f(s, x(s), x'(s)) + \int_0^s g(s,\sigma,x(\sigma))\, d\sigma \right] ds$$
$$+ \sum_{i=1}^{n+1} G'_{1s}(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\omega(s) f(s, x(s), x'(s)) + \int_0^s g(s,\sigma,x(\sigma))\, d\sigma \right] ds$$
$$- \sum_{i=1}^{n} G_1(t,t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\omega(s) f(s, x(s), x'(s)) + \int_0^s g(s,\sigma,x(\sigma))\, d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} G'_{1s}(t,t_i) I_i(x(t_i)) - \sum_{i=1}^{n} G_1(t,t_i) J_i(x(t_i)), \quad t \epsilon (t_k, t_{k+1}),$$
$$k = 0,1,2, \dots n, \qquad t_0 = 0, t_{n+1} = 1 \dots \dots \dots \dots (3.16)$$

Using Lemma (3.1) , Problem (1.1) reduces to a fixed point problem $x = Tx$, where T is given by (3.16). Thus, The problem(1.1) has a solution iff operator T has a fixed point.

Lemma 3.2 Assume that $(H_1) - (H_4)$ hold.Then $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous.

Proof:

Note that the continuity of $f, g, \omega, I_k \; and \; J_k$ together with $G_1(t,s) and \; G'_{1s}(t,s)$ ensures the continuity of T.

Let $\Omega \subset PC^1[J,R]$ be bounded. Then there exists positive constants $\mu_1, \mu_2, \mu_3 \; and \; \mu_4$ such that $|f(t,x(t),x'(t))| \leq \mu_1, |g(t,s,x(s))| \leq \mu_2, |I_k(x)| \leq \mu_3, and, |J_k(x)| \leq \mu_4 \;\; \forall x \epsilon \Omega.$

$Thus \; \forall x \epsilon \Omega,$ we have

$$|(Tx)(t)| \leq \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left[\omega(s) |f(s, x(s), x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))| d\sigma \right] ds$$
$$+ \sum_{i=1}^{n+1} |G'_{1s}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\omega(s) |f(s, x(s), x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))| d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\omega(s) |f(s, x(s), x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))| d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G'_{1s}(t,t_i)| |I_i(x(t_i))| + \sum_{i=1}^{n} |G_1(t,t_i)| |J_i(x(t_i))|$$

$$\leq \frac{\gamma\mu_1}{\Gamma(q+1)} + \mu_2\left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+2)}\right]$$
$$+ \sum_{i=1}^{n+1} |G_{1s}^{i}(t,t_i)| \left[\frac{\gamma\mu_1}{\Gamma(q+1)} + \mu_2\left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+2)}\right]\right]$$
$$+ \sum_{i=1}^{n} |G_1(t,t_i)| \left[\frac{\gamma\mu_1}{\Gamma(q)} + \mu_2\left[\frac{1}{\Gamma(q)} + \frac{1}{\Gamma(q+1)}\right]\right] + \mu_3 \sum_{i=1}^{n} |G_{1s}^{i}(t,t_i)| + \mu_4 \sum_{i=1}^{n} |G_1(t,t_i)|$$

$$\leq \gamma\mu_1\left[\frac{nc_1}{\Gamma(q)} + \frac{1+(n+1)c_2}{\Gamma(q+1)}\right] + \mu_2\left[\frac{nc_1}{\Gamma(q)} + \frac{1+(n+1)c_2+nc_1}{\Gamma(q+1)} + \frac{1+(n+1)c_2}{\Gamma(q+2)}\right]$$

$$+\mu_3 nc_2 + \mu_4 nc_1 := \mu, \qquad \dots\dots\dots\dots\dots\dots\dots (3.17)$$

Where ,

$$\gamma = \max_{t\epsilon J} \omega(t) \qquad \dots\dots\dots\dots\dots (3.18)$$

Further more, for any $t\epsilon(t_k, t_{k+1}]$

$$|(Tx)'(t)| \leq \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)}\left[\left|\omega(s)f\left(s,x(s),x'(s)\right)\right| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds$$

$$+ \sum_{i=1}^{n+1} |G_{1st}^{ii}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\left[|\omega(s)f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds$$

$$+ \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left[|\omega(s)f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds$$
$$+ \sum_{i=1}^{n} |G_{1st}^{ii}(t,t_i)||I_i(x(t_i))| + \sum_{i=1}^{n} |G'_{1t}(t,t_i)|\,|J_i(x(t_i))|$$

$$\leq \frac{\gamma\mu_1}{\Gamma(q)} + \mu_2\left[\frac{1}{\Gamma(q)} + \frac{1}{\Gamma(q+1)}\right] + \sum_{i=1}^{n+1} |G_{1st}^{ii}(t,t_i)| \left[\frac{\gamma\mu_1}{\Gamma(q+1)} + \mu_2\left[\frac{1}{\Gamma(q+1)} + \frac{1}{\Gamma(q+2)}\right]\right]$$

$$+ \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \left[\frac{\gamma\mu_1}{\Gamma(q)} + \mu_2\left[\frac{1}{\Gamma(q)} + \frac{1}{\Gamma(q+1)}\right]\right] + \mu_3 \sum_{i=1}^{n} |G_{1st}^{ii}(t,t_i)| + \mu_4 \sum_{i=1}^{n} |G'_{1t}(t,t_i)|$$

$$\leq \gamma\mu_1\left[\frac{1}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)} + \frac{nc_2}{\Gamma(q)}\right] + \mu_2\left[\frac{1+nc_2}{\Gamma(q)} + \frac{1+(n+1)c_3+nc_2}{\Gamma(q+1)} + \frac{(n+1)c_3}{\Gamma(q+2)}\right]$$

$$+\mu_3 nc_3 + \mu_4 nc_2 := \bar{\mu}\ , \qquad \dots\dots\dots\dots\dots\dots\dots (3.19)$$

On the other hand, From (3.5) for $t_1, t_2 \in (t_k, t_{k+1}]$ with $t_1 < t_2$, we have

$$|(Tx)'(t_2) - (Tx)'(t_1)| \leq \int_{t_1}^{t_2} |(Tx)''(s)|\, ds \leq \bar{\mu}(t_2 - t_1) \ \ldots\ldots\ldots\ldots \qquad (3.20)$$

It follows from $(3.17), (3.19)$ and (3.20) that T is equicontinuous on all subintervals $(t_k, t_{k+1}], k = 1,2 \ldots n.$ Thus by Arzela-Ascoli Theorem , the operator $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous.

To prove our main result,we need the following lemma.

Lemma 3.3 (Schauder Fixed point Theorem)

Let D be non empty, closed, bounded, convex subset of a banach space X , and suppose that $T: D \to D$ is completely continuous operator. Then T has fixed point $x \in D.$

IV. EXISTENCE OF SOLUTIONS

In this section we apply Lemma 3.3 to establish the existence of solution to problem (1.1).

Let us define

$$\xi_1 = \lim_{|x|+|y| \to \infty} \left(\max_{t \in J} \frac{|f(t,x,y)|}{|x| + |y|} \right), \xi_2 = \lim_{|x| \to \infty} \left(\max_{t \in J} \frac{|g(t,s,x)|}{|x|} \right)$$

$$\xi_3 = \lim_{|x| \to \infty} \frac{|I_k(x)|}{|x|}, \qquad \xi_4 = \lim_{|x| \to \infty} \frac{|J_k(x)|}{|x|}, \qquad k = 1,2, \ldots \ldots n$$

Theorem 4.1 Assume that $(H_1) - (H_4)$ hold. Suppose further that

$$\delta = \max\{\delta_1, \delta_2\} < 1 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (4.1)$$

Where

$$\delta_1 = 2\gamma\xi_1 \left[\frac{nc_1}{\Gamma(q)} + \frac{1 + (n+1)c_2}{\Gamma(q+1)} \right] + \xi_2 \left[\frac{nc_1}{\Gamma(q)} + \frac{1 + (n+1)c_2 + nc_1}{\Gamma(q+1)} + \frac{1 + (n+1)c_2}{\Gamma(q+2)} \right] \xi_3 nc_2 + \xi_4 nc_1$$

and

$$\delta_2 = 2\gamma\xi_1 \left[\frac{1 + nc_2}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)} \right] + \xi_2 \left[\frac{1 + nc_2}{\Gamma(q)} + \frac{1 + (n+1)c_3 + nc_2}{\Gamma(q+1)} + \frac{(n+1)c_3}{\Gamma(q+2)} \right] + \xi_3 nc_3 + \xi_4 nc_2$$

Where γ is defined in (3.4). Then problem (1.1) has atleast one solution $x \in PC^1[J,R] \cap C^2[J',R].$

Proof:

We shall use Schauder's fixed point theorem to prove that T has a fixed point. First, recall that the operator $T : PC^1[J,R] \to PC^1[J,R]$ is completely continuous (see the proof of lemma (3.2)).

On account of (4.1), we can choose $\xi_1' > \xi_1,\ \xi_2' > \xi_2,\ \xi_3' > \xi_3$ and $\xi_4' > \xi_4$ such that

$$\delta'_1 = 2\gamma\xi'_1\left[\frac{nc_1}{\Gamma(q)} + \frac{1+(n+1)c_2}{\Gamma(q+1)}\right] + \xi'_2\left[\frac{nc_1}{\Gamma(q)} + \frac{1+(n+1)c_2+nc_1}{\Gamma(q+1)} + \frac{1+(n+1)c_2}{\Gamma(q+2)}\right] + \xi'_3 nc_2 + \xi'_4 nc_1 \quad \ldots\ldots\ldots\ldots \quad (4.2)$$

and

$$\delta'_2 = 2\gamma\xi'_1\left[\frac{1+nc_2}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)}\right] + \xi'_2\left[\frac{1+nc_2}{\Gamma(q)} + \frac{1+(n+1)c_3+nc_2}{\Gamma(q+1)} + \frac{(n+1)c_3}{\Gamma(q+2)}\right] + \xi'_3 nc_3 + \xi'_4 nc_2 \quad \ldots\ldots\ldots\ldots \quad (4.3)$$

By the definition of ξ_1 , There exists $l > 0$ Such that

$$|f(t,x,y)| < \xi_1'(|x| + |y|), \quad \forall t \in J, |x| + |y| > l,$$

So

$$|f(t,x,y)| \le \xi_1'(|x| + |y|) + M_1, \quad \forall t \in J, x, y \in R, \ldots\ldots\ldots\ldots \quad (4.4)$$

Where

$$M_1 = \max_{t\in J, |x|+|y|\le l} |f(t,x,y)| < +\infty$$

Similarly , We have

$$|g(t,s,x)| \le \xi_2'|x| + M_2, \quad \ldots\ldots\ldots\ldots \quad (4.5)$$

$$\begin{cases} |I_k(x)| \le \xi_3'|x| + M_k, & \forall x \in R, k = 1,2\ldots n \\ |J_k(x)| \le \xi_4'|x| + \bar{M}_k, & \forall x \in R, k = 1,2\ldots n \end{cases} \quad \ldots\ldots\ldots\ldots \quad (4.6)$$

Where M_1 , M_2 , M_k , $\bar{M}_k$ are positive constants.

It follows from (3.17) and (4.4)-(4.6) that

$$\begin{aligned} |(Tx)(t)| \le & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)}\left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds \\ & + \sum_{i=1}^{n+1} |G_{1s}^i(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)}\left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds \\ & + \sum_{i=1}^{n} |G_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma\right]ds \\ & + \sum_{i=1}^{n} |G_{1s}^i(t,t_i)||I_i(x(t_i))| + \sum_{i=1}^{n} |G_1(t,t_i)||J_i(x(t_i))| \end{aligned}$$

$$\leq \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \left[\gamma(\xi_1^i(|x|+|y|)+M_1) + \int_0^s (\xi_2^i|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n+1} |G_{1s}^i(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\gamma(\xi_1^i(|x|+|y|)+M_1) + \int_0^s (\xi_2^i|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G_1(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\gamma(\xi_1^i(|x|+|y|)+M_1) + \int_0^s (\xi_2^i|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G_{1s}^i(t,t_i)|(\xi_3^i|x|+M_k) + \sum_{i=1}^{n} |G_1(t,t_i)|(\xi_4^i|x|+\bar{M}_k)$$

$$\leq \frac{1}{\Gamma(q+1)}[\gamma(2\xi_1^i\|x\|_{PC^1}+M_1)+(\xi_2^i\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+2)}\right]$$
$$+(n+1)c_2\left[\frac{1}{\Gamma(q+1)}[\gamma(2\xi_1^i\|x\|_{PC^1}+M_1)]\right.$$
$$\left.+(\xi_2^i\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+2)}\right]\right]$$
$$+nc_1\left[\frac{1}{\Gamma(q)}[\gamma(2\xi_1^i\|x\|_{PC^1}+M_1)]+(\xi_2^i\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q)}+\frac{1}{\Gamma(q+1)}\right]\right]$$
$$+nc_2(\xi_3^i\|x\|_{PC^1}+M_k)+nc_1(\xi_4^i\|x\|_{PC^1}+\bar{M}_k)$$

$$\leq \delta_1'\|x\|_{PC^1}+M^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{4.7}$$

Where δ_1' is defined by (4.2) and $M^{(1)}$ is defined by

$$M^{(1)}=\gamma M_1\left[\frac{nc_1}{\Gamma(q)}+\frac{1+(n+1)c_2}{\Gamma(q+1)}\right]+M_2\left[\frac{nc_1}{\Gamma(q)}+\frac{1+(n+1)c_2+nc_1}{\Gamma(q+1)}+\frac{1+(n+1)c_2}{\Gamma(q+2)}\right]+M_k nc_2$$
$$+\bar{M}_k nc_1$$

Similarly, from (3.11) and (4.4)-(4.6), we get

$$|(Tx)'(t)| \leq \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma \right] ds$$
$$+ \sum_{i=1}^{n+1} |G_{1st}^{ii}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\omega(s)|f(s,x(s),x'(s))| + \int_0^s |g(s,\sigma,x(\sigma))|d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G_{1st}^{ii}(t,t_i)||I_i(x(t_i))| + \sum_{i=1}^{n} |G'_{1t}(t,t_i)||J_i(x(t_i))|$$

$$\leq \int_{t_k}^{t} \frac{(t-s)^{q-2}}{\Gamma(q-1)} \left[\gamma(\xi_1'(|x|+|y|)+M_1) + \int_0^s (\xi_2'|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n+1} |G_{1st}''(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-1}}{\Gamma(q)} \left[\gamma(\xi_1'(|x|+|y|)+M_1) + \int_0^s (\xi_2'|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G'_{1t}(t,t_i)| \int_{t_{i-1}}^{t_i} \frac{(t_i-s)^{q-2}}{\Gamma(q-1)} \left[\gamma(\xi_1'(|x|+|y|)+M_1) + \int_0^s (\xi_2'|x|+M_2)d\sigma \right] ds$$
$$+ \sum_{i=1}^{n} |G_{1st}''(t,t_i)|(\xi_3'|x|+M_k) + \sum_{i=1}^{n} |G'_{1t}(t,t_i)|(\xi_4'|x|+\bar{M}_k)$$

$$\leq \frac{1}{\Gamma(q+1)}[\gamma(2\xi_1'\|x\|_{PC^1}+M_1) + (\xi_2'\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+2)}\right]$$
$$+ (n+1)c_3 \left[\frac{1}{\Gamma(q+1)}[\gamma(2\xi_1'\|x\|_{PC^1}+M_1)] \right.$$
$$\left. + (\xi_2'\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q+1)}+\frac{1}{\Gamma(q+2)}\right] \right]$$
$$+ nc_2 \left[\frac{1}{\Gamma(q)}[\gamma(2\xi_1'\|x\|_{PC^1}+M_1)] + (\xi_2'\|x\|_{PC^1}+M_2)\left[\frac{1}{\Gamma(q)}+\frac{1}{\Gamma(q+1)}\right] \right]$$
$$+ nc_3(\xi_3'\|x\|_{PC^1}+M_k) + nc_2(\xi_4'\|x\|_{PC^1}+\bar{M}_k)$$

$$\leq \delta_2'\|x\|_{PC^1} + M^{(2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4.8)$$

Where δ_2' is defined by (4.3) and $M^{(2)}$ is defined by

$$M^{(2)} = \gamma M_1 \left[\frac{1+nc_2}{\Gamma(q)} + \frac{(n+1)c_3}{\Gamma(q+1)} \right] + M_2 \left[\frac{1+nc_2}{\Gamma(q)} + \frac{1+(n+1)c_3+nc_2}{\Gamma(q+1)} + \frac{(n+1)c_3}{\Gamma(q+2)} \right] + M_k nc_3$$
$$+ \bar{M}_k nc_3$$

It follows from (4.7) and (4.8) that

$\|Tx\|_{PC^1} \leq \delta'\|Tx\|_{PC^1} + M' , \forall\, x \in PC^1[J,R]$

Where

$$\delta' = \max\{\delta_1', \delta_2'\} < 1 , \qquad M' = \max\{M^{(1)}, M^{(2)}\}.$$

Hence, we can choose a sufficiently large $r > 0$ such that $T(B_r) \subset B_r$, where

$B_r = \{x \in PC^1 : \|x\|_{PC^1} \leq r\}.$

Consequently, Lemma 3.3 implies that T has a fixed point in B_{r^*}, and the proof is complete.

Remark 4.1

Condition (4.1) is certainly satisfied if $\frac{|f(t,x,y)|}{|x|+|y|} \to 0$ uniformly in $t \in J$ as $|x| + |y| \to +\infty$, $\frac{|g(t,s,x(s))|}{|x|} \to 0$ uniformly in $s,t \in J$ as $|x| \to +\infty$, $\frac{|I_k(x)|}{|x|} \to 0$ as $|x| \to +\infty$, $\frac{|J_k(x)|}{|x|} \to 0$ as $|x| \to +\infty$, $(k = 1,2 \dots n)$.

V. EXAMPLE

To illustrate how our main result can be used in practice, we present an example
We consider the following boundary value problem:

$$\begin{cases} {}^{c}D_{0+}^{\frac{3}{2}}x(t) = b_0\, t^{\frac{1}{6}}\left[b_1\sqrt[3]{b_2 t - x + x'} - \frac{1}{20}x' - b_3 l_n(1+x^2)\right] + \frac{1}{25}\int_0^t \frac{e^{-(s-t)}x(s)}{5}\,ds, t \in J, t \neq \frac{1}{2}, \\ \Delta x|_{t_1=\frac{1}{2}} = \frac{1}{10}x\left(\frac{1}{2}\right), \qquad \Delta x'|_{t_1=\frac{1}{2}} = \frac{1}{6}x\left(\frac{1}{2}\right), \qquad \dots\dots (5.1) \\ x(0) = x(1) = 0. \end{cases}$$

Here $q = \frac{3}{2}$, $n = 1$, b_0, b_1, b_2, b_3 are positive real numbers.

This problem has atleast one solution in
$PC^1[J,R] \cap C^2[J',R]$ Where $J' = \left[0,\frac{1}{2}\right) \cup \left(\frac{1}{2},1\right]$.

Proof

It follows from (5.1) that

$\omega(t) = b_0\, t^{\frac{1}{6}}$,

$f(t,x,x') = b_1\sqrt[3]{b_2 t - x + x'} - \frac{1}{20}x' - b_3 l_n(1+x^2)$, $\quad g(t,s,x(s)) = \frac{e^{-(s-t)}x(s)}{125}$

$t_1 = \frac{1}{2}$, $\quad I_1(x) = \frac{1}{10}x$, $\quad J_1(x) = \frac{1}{6}x$, $\quad \alpha_1 = \alpha_2 = 1$, $\quad \beta_1 = \beta_2 = 0$.

From the definition of ω, f, g, I_1 and J_1, it is easy to see that $(H_1) - (H_4)$ hold. So
$\xi_1 \leq \frac{1}{20}$, $\xi_2 \leq \frac{1}{25}$, $\xi_3 \leq \frac{1}{10}$, $\xi_4 \leq \frac{1}{6}$. We have $\eta = 1$ and

$$G_1(t,s) = -\begin{cases} s(1-t), 0 \leq s \leq t \leq 1, \\ t(1-s), 0 \leq t \leq s \leq 1, \end{cases} \qquad G'_{1t}(t,s) = \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ (s-1), & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G'_{1s}(t,s) = \begin{cases} (t-1), & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1, \end{cases} \qquad G''_{1st}(t,s) = 1$$

Therefore, $c_1 = \frac{1}{4}$ $c_2 = c_3 = 1$, and therefore (4.1) is satisfied because

$$\delta_1 \leq \frac{9}{20\sqrt{\pi}} + \frac{32}{125\sqrt{\pi}} + \frac{11}{240}, \qquad \delta_2 \leq \frac{12}{30\sqrt{\pi}} + \frac{52}{125\sqrt{\pi}} + \frac{4}{15},$$

$$\delta \leq \frac{12}{30\sqrt{\pi}} + \frac{52}{125\sqrt{\pi}} + \frac{4}{15} < 1$$

Thus, our conclusion follows from theorem 4.1.

REFERENCES

[1] A.A.Kilbas,H.M.Srivastava and J.J.Trujillo, "Theory and Applications of Fractional Differential Equations", Elsevier, North-Holland Mathematics Studies, vol. 204.(2006).
[2] V.Lakshmikantham, S.Leela and J.V.Devi, "Theory of Fractional Dynamic Systems", Cambridge Scientific Publishers,Cambridge (2009).
[3] K.S.Miller, B.Ross, An Introduction to the Fractional Calculus and Differential Equations, Wiley, New York (1993).
[4] I.Podlubny, "Fractional Differential Equations", Academic Press, San Diego (1999).
[5] A.A.Elbeleze, A Kilicman and B.M.Taib, "Applications of homotopy perturbation and variational iteration methods for Fredholm integro-differential equation of fractional order", Abstr. Appl. Anal.2012, Article ID 763139 (2012).
[6] A.Kadem and A.Kilicman, "The approximate solution of fractional Fredholm integro-differential equations by variational iteration and homotopy perturbation methods", Abstr. Appl. Anal. 2012, Article ID 486193 (2012).
[7] A.Kilicman and Z.A.A.Al Zhour, "Kronecker operational matrices for fractional calculus and some applications", Appl. Math.Comput.187(1), 250-265 (2007).
[8] K.Diethelm and A.D.Freed, " On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity", Springer, Heidelberg, pp. 217-224(1999).
[9] L.Gaul, P.Klein and S.Kempfle, " Damping description involving fractional operators", Mech. Syst. Signal Process.5, 81-88 (1991).
[10] W.G.Glockle and T.F.Nonnenmacher, " A fractional calculus approach of self-similar protein dynamics" , Biophys.J.68, 46-53 (1995).
[11] R.Hilfer," Applications of Fractional Calculus in Physics", World Scientific, Singapore (2000).
[12] F.Mainardi, "Fractional calculus" , some basic problems in continuum and statistical mechanics, pp. 291-348. Springer, Wien (1997).
[13] F.Metzler, W.Schick,H.G, Kilian and T.F.Nonnenmache, "Relaxation in filled polymers, a fractional calculus approach", J.Chem. Phys.103, 7180-7186 (1995).
[14] Z.Bai and H.Lu, "Positive solutions for the boundary value problem of nonlinear fractional differential equations", J. Math.Anal. Appl.311, 495-505 (2005).
[15] R.P.Agarwal,M.Benchohra and S.Hamani, " A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions", Acta Appl. Math.109(3), 973-1033 (2010).
[16] L.Yang and H.Chen, " Nonlocal boundary value problem for impulsive differential equations of fractional order", Adv. Differ.Equ.2011, Article ID 404917 (2011).
[17] L.Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem", J. Math. Anal. Appl.162, 494-505 (1991).
[18] L.Byszewski, " Theorems about the existence and uniqueness of continuous solutions of nonlocal problem for nonlinear hyperbolic equation" , Appl. Anal.40, 173-180 (1991).
[19] L.Byszewski and H.Acka, "Existence of solutions of semilinear functional differential evolution nonlocal problems", Nonlinear Anal.34, 65-72 (1998).
[20] K.Balachandran and J.Y.Park, "Existence of mild solution of a functional integrodifferential equation with nonlocal condition" , Bull. Korean Math. Soc.38, 175-182 (2001).
[21] K.Balachandran and K.Uchiyama, " Existence of solutions of quasilinear integrodifferential equations with nonlocal condition", Tokyo J. Math.23, 203-210 (2000).
[22] K.Balachandran and D.G. Park, " Existence of solutions of quasilinear integrodifferential evolution equations in Banach spaces , Bull. Korean Math. Soc.46, 691-700 (2009).
[23] K.Balachandran and F.P.Samuel, " Existence of solutions for quasilinear delay integrodifferential equations with nonlocal condition ", Electron. J. Differ. Equ.2009(6), 1-7 (2009).
[24] G.M.N'Guerekata, " A Cauchy problem for some fractional abstract differential equation with non local conditions", Nonlinear Anal, Theory Methods Appl.70, 1873-1876 (2009).
[25] K.Balachandran and J.Y.Park, " Nonlocal Cauchy problem for abstract fractional semilinear evolution equations", Nonlinear Anal, Theory Methods Appl.71, 4471-4475 (2009).
[26] B.Ahmad, "Some existence results for boundary value problems of fractional semilinear evolution equations", Electron. J. Qual. Theory Differ. Equ.28, 1-7 (2009).
[27] K.Balachandran and J.J.Trujillo, "The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces", Nonlinear Anal, Theory Methods Appl.72, 4587-4593 (2010).
[28] M.Benchohra, J.Henderson and S.Ntouyas, " Impulsive Differential Equations and Inclusions", Hindawi Publishing Corporation, New York (2006).
[29] A.M.Samoilenko and N.A.Perestyuk, " Impulsive Differential Equations", World Scientific, Singapore (1995).
[30] Y.V.Rogovchenko, " Impulsive evolution systems, main results and new trends", Dyn. Contin. Discrete Impuls. Syst.3, 57-88 (1997).
[31] V.Lakshmikantham,D.D.Bainov and P.S.Simeonov, "Theory of Impulsive Differential Equations", World Scientific, Singapore (1989).

[32] B.Ahmad and S.Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order", Appl. Math. Comput.217(2), 480-487 (2010).
[33] X.Zhang, X.Huang and Z.Liu, "The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay", Nonlinear Anal. Hybrid Syst.4(4), 775-781 (2010).
[34] Y.Tian and Z.Bai, " Existence results for the three-point impulsive boundary value problem involving fractional differential equations", Comput. Math. Appl.59(8), 2601-2609 (2010).
[35] H.Ergoren and A.Kilicman, "Some existence results for impulsive nonlinear fractional differential equations with closed boundary conditions", Abstr. Appl. Anal.2012, Article ID 387629 (2012).
[36] J.Zhou and M.Feng, " Green's function for Sturm-Liouville-type boundary value problems of fractional order impulsive differential equations and its application", Boundary value problem. Springer Open Journal (2014).