Retrial Queue with Impatient Customers, Multi-Optional Second Phase, Bernoulli Vacation and Orbital Search

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Abstract- **Batch arrival retrial queue with multi-optional second phase is considered. Customers arrive in batches according to Poisson process. Customers are allowed to balk and renege at particular times. All the customers demand the first essential service whereas only some of them demand the second multi-optional service. After optional service, the server takes a Bernoulli vacation. On vacation completion the server may search for customers in the orbit or remain idle. The retrial time, service time and vacation time are arbitrarily distributed. The necessary and sufficient condition for the system stability is obtained. Using supplementary variable technique various steady state probabilities, expected system size, expected orbit size and availability of the server are derived. Special cases are discussed. Numerical results are presented.**

Keywords – retrial, impatience, multi optional second phase, vacation, orbital search

I. Introduction

Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time. The batch arrival process is a useful mathematical model for describing busty traffic in modern communication networks. A detailed study on batch arrival queue under both classical and constant retrial policies was done by Jain et al. (2008)..

An extensive research addressing impatience phenomena and server vacation can be found in queueing literature. Senthil Kumar and Arumuganathan (2010) considered a two phase retrial queue with impatient subscribers and general vacation time to analyse a communication protocol. Arrar et al. (2011) presented an article on the asymptotic behaviour of the M/G/1 batch arrival retrial queue with impatience phenomenon and obtained partial generating functions of the steady state joint distribution of the server state and the number of customers in the retrial group. Stithi and Djellab (2011) provided an approximation of steady state distribution of the system state of M/G/1 retrial queue with impatient customers using the principle of maximum entropy. Garg and Sanjeev Kumar (2012) obtained explicit time dependent probabilities of exact number of arrivals and departures from the orbit of a single server retrial queue with impatient customers. Sumitha et al. (2012) analysed a single server batch arrival retrial queue with balking, reneging and server vacation.

In the retrial setup, each service is preceded and followed by the server's idle time because of the ignorance of the status of the server and orbital customers by each other. Server's idle time is reduced by the introduction of search of orbital customers immediately after a service completion. Artalejo et al. (2002) considered a retrial queue in which immediately after a service completion the server searches for customers from the orbit or remains idle. Dudin et al. (2004) extended the model to a batch arrival retrial queue and performed the steady state analysis of the queueing system. Krishnamoorthy et al. (2005) analysed M/G/1 retrial queue with non persistent customers and orbital search using supplementary variable method. Chakravarthy et al. (2006) studied a multi-server retrial queueing model with orbital search. Sumitha and Udaya Chandrika (2012a) investigated a repairable M/G/1 retrial queue with Bernoulli vacation and orbital search and derived the queueing and reliability indices to predict the system behaviour. Deepak et al. (2012) obtained expected queue length of a batch arrival retrial queueing system with two types of search of customers from the orbit.

In this paper we have analysed single server batch arrival retrial queue with impatient customers, multi-optional second phase, bernoulli vacation and orbital search

II. MODEL DESCRIPTION

A single server retrial queueing system is considered. Customers arrive in batches according to Poisson process with rate λ. The batch size Y is a random variable with distribution function $P(Y = k) = C_k$, $k = 1, 2,\ldots$, probability generating function C(z) and the first two moments m_1 and m_2. If an arriving batch finds the server idle, then one of the customers obtains the service immediately and others join the orbit. If the server is blocked then the batch joins the orbit with probability p or leaves the system with probability $\bar{p}$ (=1 – p). The customer at the head of the retrial queue competes with primary customers. If a primary customer arrives first, the retrial customer cancels its attempt and returns to the retrial queue with probability q or leaves the system with probability $\bar{q}$ (= 1 – q). The retrial time of the customer is generally distributed with distribution function A(x), density function a(x), Laplace transform $A^*(s)$ and hazard rate function $\eta(x) = \frac{a(x)}{1-A(x)}$.

The server provides first essential service to all the arriving customers. The service time is generally distributed with distribution function $B_0(x)$, density function $b_0(x)$, Laplace transform $B_0^*(s)$, first two moments μ_{01}, μ_{02} and the hazard rate function $\mu_0(x) = \frac{b_0(x)}{1-B_0(x)}$.

There are M optional services. After first essential service, the customer may opt the i^{th} type of second service with probability r_i $(1 \le i \le M)$ or leave the system with probability $r_0 = 1 - \sum_{i=1}^{M} r_i$. The service time of i^{th} type optional service $(1 \le i \le M)$ is arbitrarily distributed with distribution function $B_i(x)$, density function $b_i(x)$, Laplace transform $B_i^*(s)$, the first two moments μ_{i1}, μ_{i2} and the hazard rate function $\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}$.

After completing the second optional service, the server takes a single vacation with probability ω or waits for the customers with probability $\bar{\omega}$ (= 1 – ω). The vacation time is generally distributed with distribution function V(x), density function v(x), Laplace transform $V^*(s)$, the first two moments ν_1, ν_2 and the hazard rate function $\nu(x) = \frac{v(x)}{1-V(x)}$.

After vacation completion the server searches for the customers in the orbit with probability θ or remains idle with probability $\bar{\theta}$ (= 1 – θ).

Let S(t) represent the server state 0, 1, i + 1 $(1 \le i \le M)$, M + 2 according as the server being idle, busy in first essential service, busy in i^{th} optional service, on vacation respectively. Let N(t) denote the number of customers in the orbit.

Define the following state probabilities

$I_0(t)$ is the probability that at time t the server is idle in the empty system.

$I_n(x, t)\,dx$ is the probability that at time t there are n (≥ 1) customers in the orbit, the server is idle and the elapsed retrial time is between x and x + dx.

$P_n(x, t)\,dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in first essential service and the elapsed service time is between x and x + dx.

$Q_n^i(x, t)\,dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is busy in i^{th} type optional service and the elapsed service time is between x and x + dx, $1 \le i \le M$.

$V_n(x, t)\,dx$ is the probability that at time t there are n (≥ 0) customers in the orbit, the server is on vacation and the elapsed vacation time is between x and x + dx.

III. STABILITY CONDITION

Theorem 1

Let N_n be the orbit length at the time of the departure of n^{th} customer, $n \ge 1$. Then $\{N_n, n \ge 1\}$ is ergodic if and only if

$$p \lambda m_1 \left(\mu_{01} + \sum_{i=1}^{M} (r_i \mu_{i1} + r_i \omega \nu_1)\right) + (1 - A^*(\lambda))(m_1 - 1)\left(1 - \sum_{i=1}^{M} r_i \omega \theta\right) < 1 - q\,(1 - A^*(\lambda))\left(1 - \sum_{i=1}^{M} r_i \omega \theta\right).$$

Proof

The theorem can be proved along similar lines as in Gomez-Corral (1999).

IV. STEADY STATE DISTRIBUTION

Let I_0, $I_n(x)$, $P_n(x)$, $Q_n^i(x)$ and $V_n(x)$ be the steady state probabilities of $I_0(t)$, $I_n(x, t)$, $P_n(x, t)$, $Q_n^i(x,t)$ and $V_n(x, t)$. Then the steady state equations that governs the model under consideration are given below

$$\lambda I_0 = r_0 \int_0^\infty P_0(x)\, \mu_0(x)\, dx + \overline{\omega} \int_0^\infty \sum_{i=1}^{M} Q_0^i(x)\, \mu_i(x)\, dx + \int_0^\infty V_0(x)\, \nu(x)\, dx \tag{1}$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x)) I_n(x), \qquad n \ge 1 \tag{2}$$

$$\frac{d}{dx} P_0(x) = -(p\lambda + \mu_0(x)) P_0(x) \tag{3}$$

$$\frac{d}{dx} P_n(x) = -(p\lambda + \mu_0(x)) P_n(x) + p\lambda \sum_{k=1}^{n} C_k P_{n-k}(x), \quad n \ge 1 \tag{4}$$

$$\frac{d}{dx} Q_0^i(x) = -(p\lambda + \mu_i(x)) Q_0^i(x), \quad 1 \le i \le M \tag{5}$$

$$\frac{d}{dx} Q_n^i(x) = -(p\lambda + \mu_i(x)) Q_n^i(x) + p\lambda \sum_{k=1}^{n} C_k Q_{n-k}^i(x), \; 1 \le i \le M\, ;\, n \ge 1 \tag{6}$$

$$\frac{d}{dx} V_0(x) = -(p\lambda + \nu(x)) V_0(x) \tag{7}$$

$$\frac{d}{dx} V_n(x) = -(p\lambda + \nu(x)) V_n(x) + p\lambda \sum_{k=1}^{n} C_k V_{n-k}(x), \qquad n \ge 1 \tag{8}$$

with boundary conditions

$$I_n(0) = r_0 \int_0^\infty P_n(x)\, \mu_0(x)\, dx + \overline{\omega} \int_0^\infty \sum_{i=1}^{M} Q_n^i(x)\, \mu_i(x)\, dx + \overline{\theta} \int_0^\infty V_n(x)\, \nu(x)\, dx, \; n \ge 1 \tag{9}$$

$$P_0(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x)\, \eta(x)\, dx + \theta \int_0^\infty V_1(x)\, \nu(x)\, dx \tag{10}$$

$$P_n(0) = \lambda C_{n+1} I_0 + \int_0^\infty I_{n+1}(x)\, \eta(x)\, dx + \lambda q \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)\, dx$$
$$+ \lambda \overline{q} \sum_{k=1}^{n+1} C_k \int_0^\infty I_{n-k+2}(x)\, dx + \theta \int_0^\infty V_{n+1}(x)\, \nu(x)\, dx, \qquad n \ge 1 \tag{11}$$

$$Q_n^i(0) = r_i \int_0^\infty P_n(x)\, \mu_0(x)\, dx, \qquad 1 \le i \le M;\, n \ge 0 \tag{12}$$

$$V_n(0) = \omega \int_0^\infty \sum_{i=1}^{M} Q_n^i(x)\, \mu_i(x)\, dx, \; n \ge 0 \tag{13}$$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty} \int_0^\infty I_n(x)\, dx + \sum_{n=0}^{\infty} \int_0^\infty P_n(x)\, dx + \sum_{n=0}^{\infty} \sum_{i=1}^{M} \int_0^\infty Q_n^i(x)\, dx + \sum_{n=0}^{\infty} \int_0^\infty V_n(x)\, dx = 1 \tag{14}$$

Define the probability generating functions

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x)\, z^n\, ; \qquad P(x, z) = \sum_{n=0}^{\infty} P_n(x)\, z^n\, ;$$

$$Q_i(x, z) = \sum_{n=0}^{\infty} Q_n^i(x)\, z^n, \; 1 \le i \le M \quad \text{and} \quad V(x, z) = \sum_{n=0}^{\infty} V_n(x)\, z^n$$

Multiplying equations (2) to (13) by z^n and summing over all possible values of n, we get

$$\left(\frac{\partial}{\partial x} + \lambda + \eta(x)\right) I(x, z) = 0 \tag{15}$$

$$\left(\frac{\partial}{\partial x} + p\lambda - p\lambda C(z) + \mu_0(x)\right) P(x, z) = 0 \tag{16}$$

$$\left(\frac{\partial}{\partial x} + p\lambda - p\lambda C(z) + \mu_i(x)\right) Q_i(x,z) = 0, \quad 1 \le i \le M \tag{17}$$

$$\left(\frac{\partial}{\partial x} + p\lambda - p\lambda C(z) + \nu(x)\right) V(x,z) = 0 \tag{18}$$

$$I(0,z) = r_0 \int_0^\infty P(x,z)\mu_0(x)dx + \overline{\omega} \int_0^\infty \sum_{i=1}^{M} Q_i(x,z)\mu_i(x)dx + \overline{\theta} \int_0^\infty V(x,z)\nu(x)dx - \lambda I_0 \tag{19}$$

$$P(0,z) = \frac{\lambda}{z} C(z) I_0 + \frac{1}{z}\int_0^\infty I(x,z)\eta(x)\,dx + \frac{\theta}{z}\int_0^\infty V(x,z)\,\nu(x)\,dx + \frac{\lambda q}{z} C(z)\int_0^\infty I(x,z)dx + \frac{\lambda \overline{q}}{z^2} C(z)\int_0^\infty I(x,z)dx \tag{20}$$

$$Q_i(0,z) = r_i \int_0^\infty P(x,z)\mu_0(x)\,dx, \quad 1 \le i \le M \tag{21}$$

$$V(0,z) = \omega \int_0^\infty \sum_{i=1}^{M} Q_i(x,z)\mu_i(x)\,dx \tag{22}$$

With usual procedure the solutions of the partial differential equations (15) to (18) are obtained as

$$I(x,z) = I(0,z)\, e^{-\lambda x}(1 - A(x)) \tag{23}$$
$$P(x,z) = P(0,z)\, e^{-p\lambda(1-c(z))x}(1 - B_0(x)) \tag{24}$$
$$Q_i(x,z) = Q_i(0,z)\, e^{-p\lambda(1-c(z))x}(1 - B_i(x)), \quad 1 \le i \le M \tag{25}$$
$$V(x,z) = V(0,z)\, e^{-p\lambda(1-c(z))x}(1 - V(x)) \tag{26}$$

Using equations (23) to (26) in equations (19) to (22) we get

$$I(0,z) = \lambda I_0 [z^2 - z C(z)\, B_0^*(p\lambda - p\lambda C(z))\, (r_0 + \overline{\omega} \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z)) + \omega\, \overline{\theta}\, V^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))) - z\omega\theta\, B_0^*(p\lambda - p\lambda C(z))\, V^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))] / D(z) \tag{27}$$

$$P(0,z) = \frac{\lambda I_0 [z A^*(\lambda) + C(z)(1 - A^*(\lambda))(\overline{q} + qz) - z C(z)]}{D(z)} \tag{28}$$

$$Q_i(0,z) = \lambda I_0 r_i B_0^*(p\lambda - p\lambda C(z))[z A^*(\lambda) + C(z)(1 - A^*(\lambda))(\overline{q} + qz) - zC(z)]/(z), 1 \le i \le M \tag{29}$$

$$V(0,z) = \lambda I_0 \omega [z A^*(\lambda) + C(z)(1 - A^*(\lambda))(\overline{q} + qz) - zC(z)]\, B_0^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} B_i^*(p\lambda - p\lambda C(z)) / D(z) \tag{30}$$

where

$$D(z) = [z A^*(\lambda) + C(z)(1 - A^*(\lambda))(\overline{q} + qz)]\,[B_0^*(p\lambda - p\lambda C(z)) + (r_0 + \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))\, (\overline{\omega} + \omega\overline{\theta}\, V^*(p\lambda - p\lambda C(z))))] + z\,\omega\,\theta\, B_0^*(p\lambda - p\lambda C(z))\, V^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z)) - z^2$$

Substituting the results in equations (27) to (30), I(x, z), P(x, z), $Q_i(x, z)$, V(x, z) can be expressed in terms of I_0.

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \int_0^\infty I(x,z)\,dx$$
$$= I_0(1 - A^*(\lambda))\,[z^2 - zC(z)\, B_0^*(p\lambda - p\lambda C(z))\, (r_0 + \overline{\omega} \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z)) + \omega\, \overline{\theta}\, V^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))) - z\,\omega\,\theta\, B_0^*(p\lambda - p\lambda C(z))\, V^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))]/D(z) \tag{31}$$

The partial probability generating function of the orbit size when the server is busy in essential service is

$$P(z) = \int_0^\infty P(x,z)\,dx = \frac{I_0\,[z\,A^*(\lambda)+C(z)(1-A^*(\lambda))(\bar{q}+q\,z)-z\,C(z)][1-B_0^*(p\,\lambda-p\,\lambda\,C(z))]}{D(z)(p-p\,C(z))} \quad (32)$$

The partial probability generating function of the orbit size when the server is busy in optional service is

$$Q(z) = \sum_{i=1}^{M}\int_0^\infty Q_i(x,z)\,dx = \frac{I_0\sum_{i=1}^{M} r_i\,B_0^*(p\,\lambda-p\,\lambda\,C(z))[z\,A^*(\lambda) + C(z)(1-A^*(\lambda))(\bar{q}+q\,z)-z\,C(z)][1-\sum_{i=1}^{M} B_i^*(p\,\lambda-p\,\lambda\,C(z))]}{D(z)(p-p\,C(z))} \quad (33)$$

The partial probability generating function of the orbit size when the server is on vacation is

$$V(z) = \int_0^\infty V(x,z)\,dx = \frac{I_0\,\omega\,B_0^*(p\,\lambda-p\,\lambda\,C(z))\sum_{i=1}^{M} r_i\,B_i^*(p\,\lambda-p\,\lambda\,C(z))\,[z\,A^*(\lambda) + C(z)(1-A^*(\lambda))(\bar{q}+q\,z)-z\,C(z)][1-V^*(p\,\lambda-p\,\lambda\,C(z))]}{D(z)(p-p\,C(z))} \quad (34)$$

Using the normalising condition

$I_0 + \lim_{z\to 1}[I(z) + P(z) + \sum_{i=1}^{M} Q_i(z) + V(z)] = 1$ and applying L'Hospital rule I_0 can be obtained as

$$I_0 = 1 - p\,\lambda\,m_1\,N + (1-A^*(\lambda))(\bar{q} - m_1)(1-\sum_{i=1}^{M} r_i\,\omega\,\theta)\,/\,T \quad (35)$$

where

$$T = \lambda\,(\bar{q}\,(1-A^*(\lambda)) + m_1\,\bar{p}\,A^*(\lambda))\,N + (\bar{q} + q\,A^*(\lambda)) - (1-A^*(\lambda))\,\bar{q}\sum_{i=1}^{M} r_i\,\omega\,\theta$$

and

$$N = \mu_{01} + \sum_{i=1}^{M} r_i\,\mu_{i1} + \sum_{i=1}^{M} r_i\,\omega\,\nu_1$$

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + P(z) + \sum_{i=1}^{M} Q_i(z) + V(z)$$
$$= I_0\,[(T_1(z) - z\,C(z))(1-(1-p+p\,C(z))\,T_2(z) - \omega\,\theta\,B_0^*(p\,\lambda - p\,\lambda\,C(z))\,V^*(p\lambda - p\lambda C(z))\sum_{i=1}^{M} B_i^*(p\lambda - p\,\lambda\,C(z))) + T_3(z)]\,/\,(D(z)\,(p\lambda - p\lambda\,C(z))) \quad (36)$$

where

$$T_1(z) = z\,A^*(\lambda) + C(z)\,(1-A^*(\lambda))\,(\bar{q} + q\,z)$$

$$T_2(z) = r_0\,B_0^*(p\,\lambda - p\,\lambda\,C(z)) + \bar{\omega}\,B_0^*(p\,\lambda - p\,\lambda\,C(z))\sum_{i=1}^{M} r_i\,B_i^*(p\,\lambda - p\,\lambda\,C(z)) + \omega\,\bar{\theta}\,B_0^*(p\,\lambda - p\,\lambda\,C(z))\,V^*(p\,\lambda - p\,\lambda\,C(z))\sum_{i=1}^{M} r_i\,B_i^*(p\,\lambda - p\,\lambda\,C(z))$$

and

$$T_3(z) = p\,A^*(\lambda)\,z(1-C(z))\,[T_2(z)\,C(z) - z + \omega\,\theta\,B_0^*(p\,\lambda - p\,\lambda\,C(z))\,V^*(p\,\lambda - p\,\lambda\,C(z))\sum_{i=1}^{M} r_i\,B_i^*(p\,\lambda - p\,\lambda\,C(z))]$$

The probability generating function of the number of customers in the system is

$$P_S(z) = I_0 + I(z) + z\,P(z) + z \sum_{i=1}^{M} Q_i(z) + V(z)$$

$$= I_0\,[T_3(z) + (T_1(z) - z\,C(z))\,(z - (z - p + p\,C(z))\,T_2(z) - \omega\, B_0^*(p\lambda - p\lambda C(z)) \sum_{i=1}^{M} r_i\, B_i^*(p\lambda - p\lambda C(z))((z-1)(1 - V^*(p\lambda - p\,\lambda\, C(z))) + z\,\theta\, V^*(p\,\lambda - p\,\lambda\, C(z))))] / (D(z)\,(p - p\,C(z))) \quad (37)$$

V. Performance Measures

- The steady state probability that the server is idle in non empty system is

$$I = I(1)$$
$$= (1 - A^*(\lambda))\,(p\,\lambda\, m_1\, N + m_1(1 - \sum_{i=1}^{M} r_i\,\omega\,\theta) - 1) / T \quad (38)$$

- The steady state probability that the server is busy in the first essential service is

$$P = P(1)$$
$$= \lambda\,\mu_{01}\,[m_1\, A^*(\lambda) + (1 - A^*(\lambda))\,\overline{q}\,] / T \quad (39)$$

- The steady state probability that the server is busy in the optional service is

$$Q = Q(1)$$
$$= \lambda\,[m_1\, A^*(\lambda) + (1 - A^*(\lambda))\,\overline{q}\,] \sum_{i=1}^{M} r_i\,\mu_{i1} / T \quad (40)$$

- The steady state probability that the server is on vacation is

$$V = V(1)$$
$$= \omega\,\lambda\,\nu_1\,(m_1\, A^*(\lambda) + (1 - A^*(\lambda))\,\overline{q}) \sum_{i=1}^{M} r_i \;/ T \quad (41)$$

- Expected number of customers in the system when the server is idle in the non-empty system is

$$L_I = \lim_{z\to 1} \frac{d}{dz} I(z)$$
$$= (N_3\, N_2 - N_1\, N_4) / (2\, N_3^2) \quad (42)$$

where

$$N_1 = (1 - A^*(\lambda))\, I_0\,(1 - p\,\lambda\, m_1\, N - m_1\,(1 - \omega\,\theta \sum_{i=1}^{M} r_i))$$

$$N_2 = (1 - A^*(\lambda))\, I_0\,[2 - (p\,\lambda\, m_2\, N + p^2\lambda^2\, m_1^2\,(\mu_{02} + \sum_{i=1}^{M} r_i\,\mu_{i2} + \omega\,\nu_2 \sum_{i=1}^{M} r_i) + 2\,p^2\lambda^2\, m_1^2 \sum_{i=1}^{M} r_i\,(\mu_{01}\,\mu_{i1} + \omega\,\mu_{i1}\,\nu_1 + \omega\,\nu_1\,\mu_{01})) + 2\,p\,\lambda\, m_1\, N\,(1 + m_1) - (2\,m_1 + m_2)\,(1 - \omega\,\theta \sum_{i=1}^{M} r_i)]$$

$$N_3 = p\,\lambda\, m_1\, N - (1 - A^*(\lambda))\,(1 - q - m_1)\,(1 - \omega\,\theta \sum_{i=1}^{M} r_i) - 1$$

$$N_4 = p\,\lambda\, m_2\, N + p^2\lambda^2\, m_1^2\,(\mu_{02} + \sum_{i=1}^{M} r_i\,\mu_{i2} + \omega\,\nu_2 \sum_{i=1}^{M} r_i) + 2 \sum_{i=1}^{M} r_i\, p^2\lambda^2\, m_1^2\,(\mu_{01}\,\mu_{i1} + \omega\,\nu_1\, b_{i1} + \omega\, b_{01}\,\nu_1) + 2\,(A^*(\lambda) + (1 - A^*(\lambda))\,(q + m_1))\,[p\,\lambda\, m_1\,((\mu_{01} + \sum_{i=1}^{M} r_i\,\mu_{i1} + \omega\,\nu_1 \sum_{i=1}^{M} r_i) - \omega\,\theta \sum_{i=1}^{M} r_i\,(\mu_{01} + \mu_{i1} + \nu_1))] + 2\,\omega\,\theta \sum_{i=1}^{M} r_i\, p\,\lambda\, m_1\,(\mu_{01} + \mu_{i1} + \nu_1) + (1 - A^*(\lambda))\,(2\,q\,m_1 + m_2)\,(1 - \omega\,\theta \sum_{i=1}^{M} r_i) - 2$$

- Expected number of customers in the system when the server is busy in essential service is

$$L_P = \lim_{z\to 1} \frac{d}{dz} P(z)$$
$$= (N_7\, N_6 - N_5\, N_8) / (3\, N_7^2) \quad (43)$$

where

$$N_5 = -I_0\,(2\,p\,\lambda\, m_1\,\mu_{01})\,(A^*(\lambda) + (1 - A^*(\lambda))\,(m_1 + q) - (m_1 + 1))$$

$N_6 = -3 I_0 [(A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q) - (m_1 + 1))(p \lambda m_2 \mu_{01} + p^2 \lambda^2 m_1^2 \mu_{02}) + p \lambda m_1 \mu_{01} ((1 - A^*(\lambda))(2 q m_1 + m_2) - (m_2 + 2 m_1))]$

$N_7 = -2 p m_1 N_3$

$N_8 = -3 p m_1 N_4 - 3 p m_2 N_3$

- Expected number of customers in the system when the server is busy in optional service is

$$L_Q = \lim_{z \to 1} \frac{d}{dz} \sum_{i=1}^{M} Q_i(z) = (N_7 N_{10} - N_9 N_8) / (3 N_7^2) \qquad (44)$$

where

$N_9 = -2 I_0 p \lambda m_1 \sum_{i=1}^{M} r_i \mu_{i1} (A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q) - (m_1 + 1))$

$N_{10} = -3 I_0 [(A^*(\lambda) + (1 - A^*(\lambda))(m_1 + q) - (m_1 + 1)) \sum_{i=1}^{M} r_i (p \lambda m_2 \mu_{i1} + p^2 \lambda^2 m_1^2 \mu_{i2} + 2 p \lambda m_1 \mu_{01} + 2 p \lambda m_1 \mu_{i1}) + p \lambda m_1 \sum_{i=1}^{M} r_i \mu_{i1} ((1 - A^*(\lambda))(2 q m_1 + m_2) - (m_2 + 2 m_1))]$

- Expected number of customers in the system when the server is on vacation is

$$L_V = \lim_{z \to 1} \frac{d}{dz} V(z) = (N_7 N_{12} - N_{11} N_8) / (3 N_7^2) \qquad (45)$$

where

$N_{11} = -2 I_0 \omega \sum_{i=1}^{M} r_i (p \lambda m_1 \nu_1 (A^*(\lambda) + (1 - A^*(\lambda))(q + m_1) - (m_1 + 1)))$

$N_{12} = -3 I_0 \omega \sum_{i=1}^{M} r_i [(A^*(\lambda) + (1 - (A^*(\lambda)(q + m_1) - (m_1 + 1))(p \lambda m_2 \nu_1 + p^2 \lambda^2 m_1^2 \nu_2 + 2 p^2 \lambda^2 m_1^2 \nu_1 (\mu_{01} + \mu_{i1})) + p \lambda m_1 \nu_1 ((1 - A^*(\lambda))(2 q m_1 + m_2) - (m_2 + 2 m_1))]$

- The mean number of customers in the orbit :
  The mean number of customers in the orbit L_q is given by

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z) = \frac{Dr''(1)\ Nr'''(1) - Nr''(1)\, Dr'''(1)}{3 Dr''(1)^2} \qquad (46)$$

where Nr(z) and Dr(z) be the numerator and denominator of $P_q(z)$

$Nr''(1) = 2I_0[p\lambda m_1 N(\bar{q}(1 - A^*(\lambda)) + \bar{p}\, m_1 A^*(\lambda)) + p m_1(1 - q(1 - A^*(\lambda))) - \omega \theta p m_1 (\bar{q}(1 - A^*(\lambda))) \sum_{i=1}^{M} r_i]$

$Nr'''(1) = 3 I_0 [(g_2 + 2p m_1 g_1 + p m_2 (1 - \sum_{i=1}^{M} r_i \omega \theta))(\bar{q}(A^*(\lambda) - 1) - m_1 A^*(\lambda)) + (p \lambda m_1 N + p m_1 (1 - \sum_{i=1}^{M} r_i \omega \theta))\ (2 q m_1 (1 - A^*(\lambda)) - 2 m_1 - m_2 A^*(\lambda)) + g_3]$

$g_1 = p \lambda m_1 [\mu_{01} + \sum_{i=1}^{M} r_i \mu_{i1} + \sum_{i=1}^{M} r_i \omega \nu_1 - \sum_{i=1}^{M} r_i \omega \theta (\mu_{01} + \mu_{i1} + \nu_1)]$

$g_2 = p \lambda m_2 N + p^2 \lambda^2 m_1^2 (\mu_{02} + \sum_{i=1}^{M} r_i \mu_{i2} + \omega \nu_2 \sum_{i=1}^{M} r_i + 2 \mu_{01} \sum_{i=1}^{M} r_i \mu_{i1} + 2 \omega \nu_1 \sum_{i=1}^{M} r_i \mu_{i1} + 2 \omega \mu_{01} \nu_1 \sum_{i=1}^{M} r_i)$

$g_3 = p A^*(\lambda) [(p \lambda m_1 N + m_1 - 1)(2m_1 + m_2) + m_1 (2 p\lambda m_1^2 N - 2 \sum_{i=1}^{M} r_i \omega \theta p \lambda m_1^2 (\mu_{01} + \mu_{i1} + \nu_1) + m_2 + g_2)]$

$Dr''(1) = 2 p m_1[p \lambda m_1 N - (1 - A^*(\lambda))(\bar{q} - m_1)(1 - \sum_{i=1}^{M} r_i \omega \theta) - 1]$

$$Dr'''(1) = -3\,[p\, m_2\, (p\lambda m_1\, N-(1-A^*(\lambda))\,(\overline{q}-m_1)\,(1-\sum_{i=1}^{M} r_i\,\omega\,\theta)-1)+p\, m_1\,[p^2\,\lambda^2\, m_1^2\,(\mu_{02}+\sum_{i=1}^{M} r_i\,\mu_{i2}+\sum_{i=1}^{M} r_i\,\omega\,\nu_2)$$
$$+\,p\,\lambda\, m_2\, N+\sum_{i=1}^{M} 2\, r_i\, p^2\,\lambda^2\, m_1^2\,(\mu_{01}\mu_{i1}+\omega\mu_{i1}\nu_1+\omega\mu_{01}\nu_1)+2\,((A^*(\lambda)+(1-A^*(\lambda))\,(q+m_1))\,p\,\lambda\, m_1\, N-$$
$$\sum_{i=1}^{M} r_i\,\omega\,\theta(\mu_{01}+\mu_{i1}+\nu_1))]+2\sum_{i=1}^{M} r_i\,\omega\,\theta\, p\,\lambda\, m_1(\mu_{01}+\mu_{i1}+\nu_1)+(1-A^*(\lambda))\,(2q\, m_1+m_2)\,(1-\sum_{i=1}^{M} r_i\,\omega\,\theta)-2]$$

- The mean number of customers in the system under steady state is

$$L_S = L_q + P + Q \qquad (47)$$

- Availability of the server under steady state is

$$A = I_0 + I + P + Q$$

$$= \frac{I_0[(\overline{q}\,(1-A^*(\lambda))+\lambda\overline{p}\, m_1\, A^*(\lambda))\,(\mu_{01}+\sum_{i=1}^{M} r_i\,\mu_{i1}) + (\overline{q}+q\, A^*(\lambda))-\overline{q}\,(1-A^*(\lambda))\,\omega\,\theta\sum_{i=1}^{M} r_i - A^*(\lambda)\,\omega\, p\,\lambda\, m_1\,\nu_1\sum_{i=1}^{M} r_i]}{1-p\,\lambda\, m_1\, N + (1-A^*(\lambda))\,(\overline{q}-m_1)\,(1-\omega\,\theta\sum_{i=1}^{M} r_i)}$$

$$= [(\overline{q}\,(1-A^*(\lambda))+\overline{p}\,\lambda\, m_1\, A^*(\lambda))\,(\mu_{01}+\sum_{i=1}^{M} r_i\,\mu_{i1})+(\overline{q}+q\, A^*(\lambda))$$
$$-\,\overline{q}\,(1-A^*(\lambda))\,\omega\,\theta\sum_{i=1}^{M} r_i - A^*(\lambda)\,\omega\, p\,\lambda\, m_1\,\nu_1\sum_{i=1}^{M} r_i\,]\,/\,T \qquad (48)$$

VI. SPECIAL CASES

Case (i)

Suppose that $C(z) \rightarrow z$, $q = 1$, $M = 1$, $\theta = 0$ (single arrival, no reneging, single optional service, no orbital search) then the model under study reduces to two Phase M/G/1 retrial queue with balking and vacation. In this case

$$I(z) = \frac{z\, I_0\,(1-A^*(\lambda))\,(1-T_4(z))}{(A^*(\lambda)+z(1-A^*(\lambda)))\,T_4(z)-z}$$

$$P(z) = \frac{I_0\, A^*(\lambda)\,(1-B_0^*(p\lambda-p\lambda(z)))}{p[(A^*(\lambda)+z(1-A^*(\lambda)))\,T_4(z)-z]}$$

$$Q(z) = \frac{I_0\, A^*(\lambda)\, B_0^*(p\lambda-p\lambda(z))\,(1-B_1^*(p\lambda-p\lambda(z)))}{p[(A^*(\lambda)+z(1-A^*(\lambda)))\,T_4(z)-z]}$$

$$V(z) = \frac{I_0\, A^*(\lambda)\,\omega\, B_0^*(p\lambda-p\lambda(z))\, B_1^*(p\lambda-p\lambda(z))\,(1-V^*(p\lambda-p\lambda(z)))}{p[(A^*(\lambda)+z(1-A^*(\lambda)))\,T_4(z)-z]}$$

$$I_0 = \frac{A^*(\lambda)-p\,\lambda\,(\mu_{01}+\mu_{11}+\omega\nu_1)}{A^*(\lambda)\,[\overline{p}\,\lambda\,(\mu_{01}+\mu_{11}+\omega\nu_1)+1]}$$

$$P_q(z) = \frac{I_0\, A^*(\lambda)\,[(T_4(z)-1)+p(z-T_4(z))]}{p[z-(A^*(\lambda)+(1-A^*(\lambda))z)\,T_4(z)]}$$

where $T_4(z) = B_0^*(p\lambda - p\lambda z)\ B_1^*(p\lambda - p\lambda z)\,[\omega\, V^*(p\lambda - p\lambda z) + \overline{\omega}\,]$

The results coincide with those of Senthil Kumar and Arumuganathan (2010).

Case (ii)

If $C(z) \rightarrow z$, $p = q = 1$, $M = 1$, $\omega = 0$, $\theta = 0$ (single arrival, no balking and reneging, single optional service, no vacation, no orbital search) then the model reduces to M/G/1 retrial queue with two phases of service. In this case

$$I(z) = \frac{z\,[A^*(\lambda)-\lambda(\mu_0+r_1\,\mu_1)]\,[1-T_5(z)]\,[1-A^*(\lambda)]}{A^*(\lambda)\,[T_5(z)\,(A^*(\lambda)\,(1-z)+z)-z]}$$

$$P(z) = \frac{[A^*(\lambda)-\lambda(\mu_0+r_1\,\mu_1)]\,(1-B_0^*(\lambda-\lambda z))}{T_5(z)[A^*(\lambda)\,(1-z)+z]-z}$$

$$Q(z) = \frac{[A^*(\lambda)-\lambda(\mu_0+r_1\,\mu_1)]\, r_1\, B_0^*(\lambda-\lambda z)\,(1-B_1^*(\lambda-\lambda z))}{T_5(z)[A^*(\lambda)\,(1-z)+z]-z}$$

$$I_0 = \frac{A^*(\lambda)-\lambda\,(\mu_0+r_1\,\mu_1)}{A^*(\lambda)}$$

$$P_q(z) = \frac{I_0\, A^*(\lambda)\,(1-z)}{T_5(z)\,[A^*(\lambda)\,(1-z)+z] - z}$$

where $T_5(z) = B_0^*(\lambda - \lambda z)\,[r_0 + r_1 \; B_1^*(\lambda - \lambda z)]$

The above results coincide with the results given in Choudhury (2009).

VII. Numerical Results

In order to explore the effect of various system parameters on the performance measures, numerical evaluation is carried out and the results are displayed in graphs.

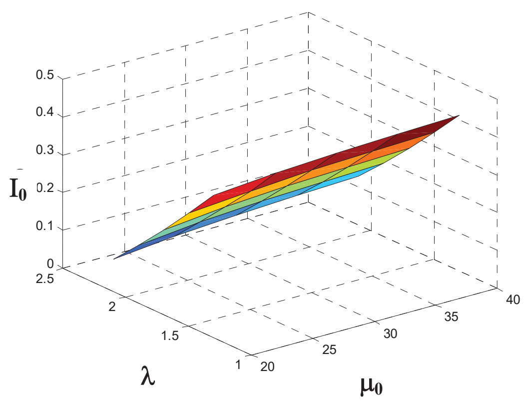
Assume that the retrial time, essential service time, optional service time and vacation time follow exponential distribution with respective rates η, μ_0, μ_i ($1 \le i \le M$) and ν. For computation we choose arbitrarily the parameters $\lambda = 1.5$, $p = 0.7$, $q = 0.7$, $\omega = 0.5$, $\theta = 0.7$, $\eta = 30$, $M = 3$, $r_0 = 0.4$, $r_1 = 0.3$, $r_2 = 0.2$, $r_3 = 0.1$, $\mu_0 = 15$, $\mu_1 = 2$, $\mu_2 = 2$, $\mu_3 = 3$, $\nu = 7$, $C_1 = 0.5$ and $C_2 = 0.5$ that satisfy stability condition.

The combined effect of λ and μ_0 on I_0, I, P, Q, V and L_S are presented in Fig.1 (a) to (f). From the figures it is observed that

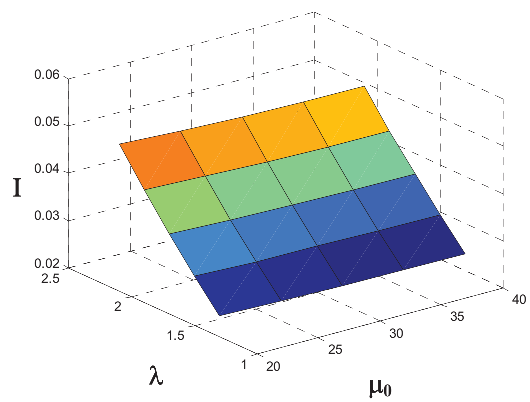
- I_0 is a decreasing function of λ and an increasing function of μ_0.
- I and P are increasing functions of λ and decreasing functions of μ_0.
- Q and V are increasing functions of λ and μ_0.
- L_S is an increasing function of λ and decreasing function of μ_0.

The combined effect of η and ω on the performance measures are displayed in Fig.2 (a) to (f) reveals that

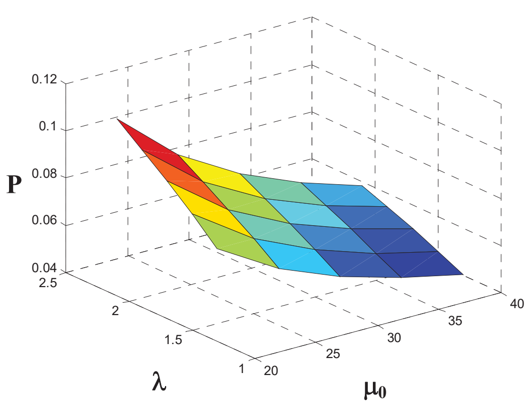
- I_0, P, Q and V are increasing functions of η. I and L_S are decreasing functions of η.
- V and L_S are increasing functions of ω. I_0, I, P and Q are decreasing functions of ω.



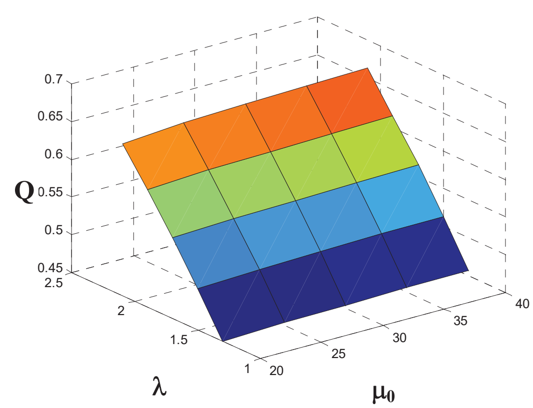
(a) I_0 versus (λ, μ_0)



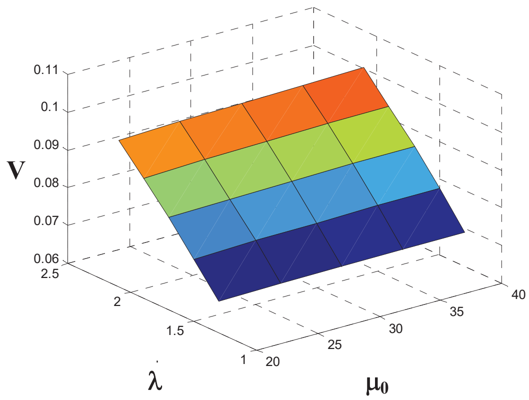
(b) I versus (λ, μ_0)



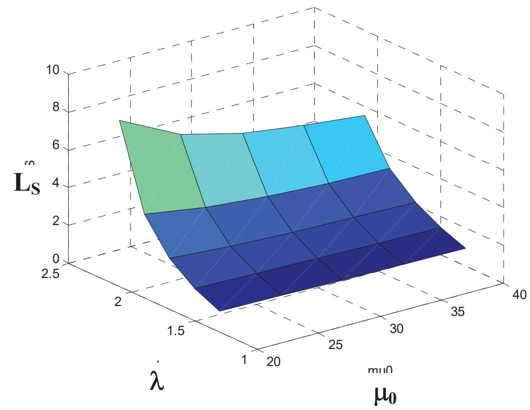
(c) P versus (λ, μ_0)



(d) Q versus (λ, μ_0)

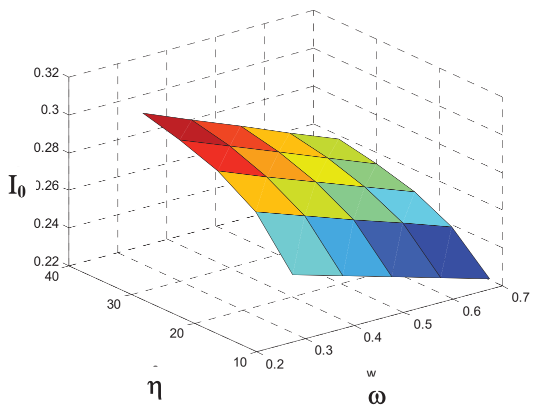


(e) V versus (λ, μ_0)

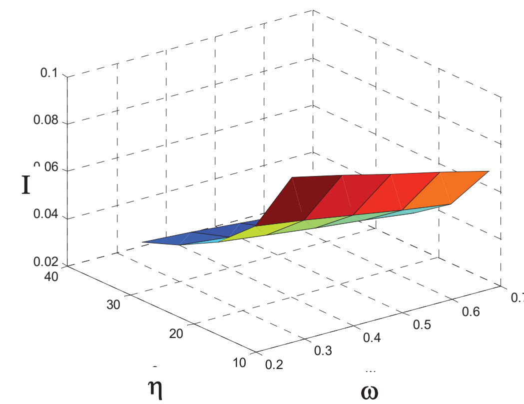


(f) L_S versus (λ, μ_0)

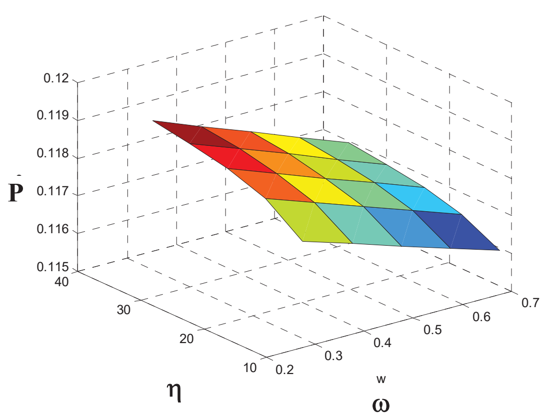
Fig. 1 Influence of λ and μ_0 on the Performance Measures



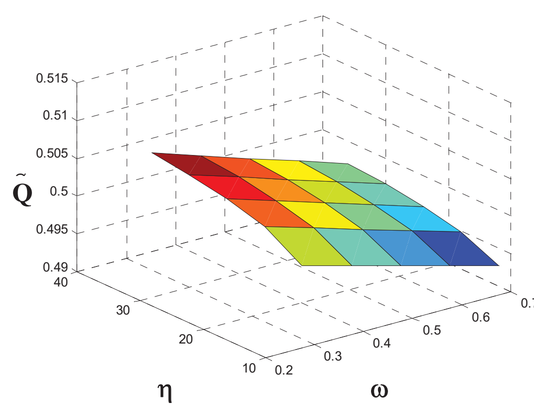
(a) I_0 versus (η, ω)



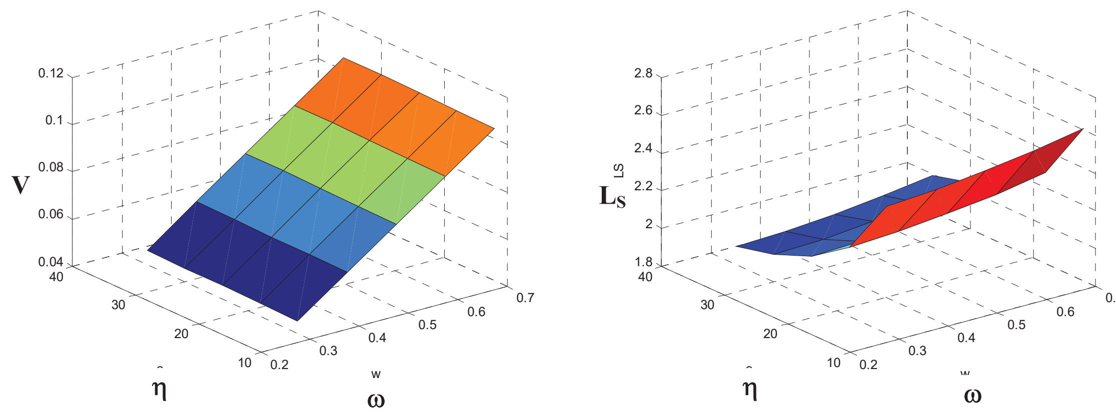
(b) I versus (η, ω)



(c) P versus (η, ω)



(d) Q versus (η, ω)



e) V versus (η, ω) (f) L_S versus (η, ω)

Fig. 2 Influence of η and ω on the Performance Measures

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