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Incompletness of Lukasiewicz and Gödel logics

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Abstract. Every k-valued logic is defined by a model such that every logical formula has a truth-table. We suppose that every truth-table of the model must have a logical formula, too. Then the logic is complete. This is a very strong condition since every model has k^{k^n} n-ary truth-tables. Nevertheless complete logics exist. But Lukasiewicz and Gödel logics are not complete.

I. LOGIC, ALGEBRA AND COMPLETENESS

Every logic is a theory and every theory has a model. The model of a logic is an algebra. We name objects of a logic and its model *formulas and functions* respectively.

A formula is a propositional variable or several propositional variables separated by connectivities $\vee$, $\wedge$ and $\neg$. The connective $\neg$ usually is replaced by a diacritic line. We will denote propositional variables by p, p_1, p_2,... .

Every function has a truth-table. We denote variables of functions by x, x_1, x_2,... . The range of functions for k-valued logics is the set $\{0,1,...,k-1\}$.

Every formula is interpreted by a function. It means that every formula has a truth-table. We call a logic *complete*, if every truth-table of the algebra has a formula of the logic.

Complete logics exist. For example, Post logic P_k is complete [1]. Post proved this.

II. LUKASIEWICZ LOGIC

We shall prove that Lukasiewicz logic L_k [2] is incomplete.

The logic has the next interpretations of its formulas.

Definition *The formulas* $p_1 \vee p_2$, $p_1 \wedge p_2$, *and* $\overline{p}$ *are interpreted by the computable functions* $\max(x_1, x_2)$, $\min(x_1, x_2)$, *and* $k-1-x$.

By definition operations $\vee$ and $\wedge$ are idempotent, associative, commutative, absorbing and distributive. The operation $\neg$ has the next property: $\overline{\overline{p}} = p$

Theorem *Lukasiewicz logic is incomplete.*

Proof. Every algebra of a logic has k^k one-ary functions. We shall prove that only 4 of them have formulas in the logic. These formulas are p, $\overline{p}$, $p \vee \overline{p}$, $p \wedge \overline{p}$. Indeed, using operations $\vee$, $\wedge$ and $\neg$ we have no new formula:

$$p \vee (p \wedge \overline{p}) = p,\ \overline{p} \vee (p \wedge \overline{p}) = \overline{p},\ p \wedge (p \vee \overline{p}) = p,\ \overline{p} \wedge (p \vee \overline{p}) = \overline{p},\ \overline{p \vee \overline{p}} = p \wedge \overline{p},\ \overline{p \wedge \overline{p}} = p \vee \overline{p}$$

III. GÖDEL LOGIC

Gödel logic G_k [3] has the next interpretations of logical operations.

Definition *The formulas* $p_1 \vee p_2$, $p_1 \wedge p_2$, *and* $\overline{p}$ *are interpreted by the computable functions* $\max(x_1, x_2)$, $\min(x_1, x_2)$, *and* $(k-1)(1-\mathrm{sg}(x))$, *where* $\mathrm{sg}(x) = 0$ *if* $x = 0$ *and* $\mathrm{sg}(x) = 1$ *if* $x \neq 0$.

By definition $\overline{x}=0$ if $x\neq 0$ and $\overline{0}=k-1$. Hear $\overline{x}=(k-1)(1-sg(x))$. The other operations do not change.

Theorem *Gödel logic is incomplete.*

Proof. We shall prove that only 6 of one-ary functions have formulas in the logic. These formulas are p, $\overline{p}$, $\overline{\overline{p}}$, $p\vee\overline{p}$, $p\wedge\overline{p}$, $\overline{p\wedge\overline{\overline{p}}}$.

Indeed, using $\vee$ we have no new formula:

$$p\vee\overline{\overline{p}}=\overline{\overline{p}},\quad p\vee(p\vee\overline{p})=p\vee\overline{p},\quad p\vee(p\wedge\overline{p})=p,\quad p\vee\overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}},$$

$$\overline{p}\vee\overline{\overline{p}}=\overline{p\wedge\overline{\overline{p}}},\quad \overline{p}\vee(p\vee\overline{p})=p\vee\overline{p},\quad \overline{p}\vee(p\wedge\overline{p})=\overline{p},\quad \overline{p}\vee\overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}},$$

$$\overline{\overline{p}}\vee(p\vee\overline{p})=\overline{p\wedge\overline{\overline{p}}},\quad \overline{\overline{p}}\vee(p\wedge\overline{p})=\overline{\overline{p}},\quad \overline{\overline{p}}\vee\overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}},$$

$$(p\vee\overline{p})\vee(p\wedge\overline{p})=p\vee\overline{p},\quad (p\vee\overline{p})\vee\overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}},\quad (p\wedge\overline{p})\vee\overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}}$$

Using $\wedge$ we have no new formula, too:

$$p\wedge\overline{\overline{p}}=p,\quad p\wedge(p\vee\overline{p})=p,\quad p\wedge(p\wedge\overline{p})=(p\wedge\overline{p}),\quad p\wedge\overline{p\wedge\overline{\overline{p}}}=p$$

$$\overline{p}\wedge\overline{\overline{p}}=p\wedge\overline{p},\quad \overline{p}\wedge(p\vee\overline{p})=\overline{p},\quad \overline{p}\wedge(p\wedge\overline{p})=p\wedge\overline{p},\quad \overline{p}\wedge\overline{p\wedge\overline{\overline{p}}}=\overline{p},$$

$$\overline{\overline{p}}\wedge(p\vee\overline{p})=p,\quad \overline{\overline{p}}\wedge(p\wedge\overline{p})=p\wedge\overline{p},\quad \overline{\overline{p}}\wedge\overline{p\wedge\overline{\overline{p}}}=\overline{\overline{p}},$$

$$(p\vee\overline{p})\wedge(p\wedge\overline{p})=p\wedge\overline{p},\quad (p\vee\overline{p})\wedge\overline{p\wedge\overline{\overline{p}}}=p\vee\overline{p},\quad (p\wedge\overline{p})\wedge\overline{p\wedge\overline{\overline{p}}}=p\wedge\overline{p}$$

And using $\neg$ we again have no new formula:

$$\overline{\overline{\overline{p}}}=\overline{p},\quad \overline{p\vee\overline{\overline{p}}}=p\wedge\overline{p},\quad \overline{p\wedge\overline{\overline{p}}}=\overline{p\wedge\overline{\overline{p}}},\quad \overline{\overline{p\wedge\overline{\overline{p}}}}=p\wedge\overline{p}$$

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