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Some Unified Finite Integrals involving Generalized Associated Legendre Polynomials

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Abstract- In this paper four unified finite integrals are established. The first two of these integrals involve the product of a general class of a multivariable polynomials, $\bar{H}$ function and generalized associated Legendre function of second kind with arguments of the form $(x-a-1)^{-\frac{n}{2}}(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(g+xf)^{\delta}$. In third and fourth integrals general class of a multivariable polynomials are replaced with a generalized polynomial set in first and second integral. Some special cases and applications are also discussed. Since functions and polynomials occurring in these integrals are general in nature, these results provide interesting unifications and extensions of a large number of new and known results.

Keywords – $\bar{H}$ function, general class of a multivariable polynomials, generalized associated Legendre function, generalized polynomial set, fractional operator.

I. INTRODUCTION

A large number of integral formulae involving different types of special functions have been developed by many authors. Garg and Mittal [1] obtained an interesting unified integral involving Fox H-function. Considering the work of Garg and Mittal [1], Ali [2] gave three interesting unified integrals involving the hypergeometric function ${}_1F_2$. By using Ali's method [2] Choi and Agarwal [3] presented two generalized integral formulas involving the Bessel function of the first kind , which are expressed in terms of the generalized (Wright) hypergeometric functions.

Agarwal [4] study some new unfied integral formulae associated with the $\bar{H}$ -function. Each of these formulae involves a product of the $\bar{H}$ -function and Srivastava polynomials with essentially arbitrary coefficients. They evaluated the formulae in terms of $\psi(z)$ [logarithmic derivative of $\Gamma(z)$]. Recently Chouhan and Khan [5] presents two new unified integral formulae involving the Fox H-function and M-Series. These results were expressed in terms of the H function.

II. SOME DEFINITIONS

A. *Riemann-Liouville fractional calculus operator*

The Riemann-Liouville fractional calculus operator of order $-\mu$ defined by Miller and Ross [6] as

$$
{}_a\mathrm{D}_z^{-\mu}\left[f(z)\right] = \begin{cases} \dfrac{1}{\Gamma(\mu)}\displaystyle\int_a^z (z-t)^{\mu-1} f(t)\,\mathrm{dt}, \ \mathrm{Re}(\mu)>0 \\ \dfrac{\mathrm{d}^m}{\mathrm{dz}^m}\,{}_a\mathrm{D}_z^{-\mu-m}\left[f(z)\right], \ -m<\mathrm{Re}(\mu)\le 0,\ m\in\mathrm{N} \end{cases} \quad \ldots (2.1)
$$

where *m* is a positive integer and the integral exists.

B. $\bar{H}$ *- function*

A more general function known as $\bar{\mathrm{H}}$ -function was introduced by Inayat-Hussain [7] in the following form

$$\bar{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}[z] = \bar{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{\mathrm{N+1,P}}\\(b_j,\beta_j)_{1,\mathrm{M}},(b_j,\beta_j;\mathrm{B}_j)_{\mathrm{M+1,Q}}\end{matrix}\right.\right] = \frac{1}{2\pi \mathrm{i}}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)\,z^{\xi}\,\mathrm{d}\xi \quad \text{… (2.2)}$$

Where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{\mathrm{M}}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{\mathrm{N}}\{\Gamma(1-a_j+\alpha_j\xi)\}^{\mathrm{A}_j}}{\prod_{j=\mathrm{M}+1}^{\mathrm{Q}}\{\Gamma(1-b_j+\beta_j\xi)\}^{\mathrm{B}_j}\prod_{j=\mathrm{N}+1}^{\mathrm{P}}\Gamma(a_j-\alpha_j\xi)} \quad \text{… (2.3)}$$

and $i = \sqrt{-1}$. Here a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameters, $\alpha_j \geq 0\,(j = 1,...,\mathrm{P})$ and $\beta_j \geq 0\,(j = 1,...,\mathrm{Q})$ and the exponents A_j (j = 1,…,N) and B_j (j = N+1,…,Q) can take any non-integer values .

C. Multivariable Polynomials

The second general class of multivariable polynomials given by Srivastava [8] is defined and represented in the following modified form

$$\mathrm{S}_{\mathrm{V}_1,...,\mathrm{V}_r}^{\mathrm{U}_1,...,\mathrm{U}_r}[x_1,...,x_r] = \sum_{k_1=0}^{[\mathrm{V}_1/\mathrm{U}_1]}...\sum_{k_r=0}^{[\mathrm{V}_r/\mathrm{U}_r]}(-\mathrm{V})_{\mathrm{U}_1k_1}...(-\mathrm{V}_r)_{\mathrm{U}_rk_r}\cdot \mathrm{A}\,(\mathrm{V}_1k_1;...;\mathrm{V}_rk_r)\,\frac{x_1^{k_1}}{k_1!}...\frac{x_r^{k_r}}{k_r!} \quad \text{… (2.4)}$$

Where $\mathrm{V}_i = 0,1,2,...(i = 1,...,r)$, $\mathrm{U}_1,...,\mathrm{U}_r$ are arbitrary positive integers, the coefficients $\mathrm{A}\,[\mathrm{V}_1,k_1;...;\mathrm{V}_r,k_r]$ being arbitrary constants, real or complex.

D. Generalized Polynomial Set

The generalized polynomial set $\mathrm{S}_n^{\alpha,\beta;\tau}[x]$ is defined by the following Rodrigues type formula [9]

$$\mathrm{S}_{\mathrm{N}}^{\alpha,\beta,\tau}[x] = (\mathrm{A}x+\mathrm{B})^{-\alpha}(1-\tau x^r)^{\beta/\tau}\,\mathrm{T}_{k,\ell}^{\mathrm{N}}\left[(\mathrm{A}x+\mathrm{B})^{\alpha+b\mathrm{N}}(1-\tau x^r)^{\frac{\beta}{\tau}+\alpha\mathrm{N}}\right] \quad \text{… (2.5)}$$

with the differential operator being defined as

$$\mathrm{T}_{k,\ell} = x^{\ell}(k + x\,\mathrm{D}_x)$$

where $\mathrm{D}_x \equiv \mathrm{d}/\mathrm{d}x$

Raizada [9] presented $\mathrm{S}_n^{\alpha,\beta;\tau}[x]$ in the following series form

$$\mathrm{S}_n^{\alpha,\beta,\tau}[x] = \sum_{b_1,b_2,a_1,a_2}\theta(b_1,b_2,a_1,a_2)\,x^{\mathrm{R}}(1-\tau x^t)^{sn-a_1} \quad \text{… (2.6)}$$

Where

$$\theta(b_1,b_2,a_1,a_2) = \frac{\mathrm{B}^{bn}\,\ell^n(-\tau)^{a_1}(-1)^{b_1}(-a_1)_{a_2}(-b_1)_{b_2}(\alpha)_{b_1}(-\alpha-bn)_{a_2}}{a_1!a_2!b_1!b_2!(1-\alpha-b_1)_{b_2}}\left(\frac{-\beta}{\tau}-sn\right)_{a_1}\left(\frac{b_2+k+ta_2}{\ell}\right)_n\left(\frac{\mathrm{A}}{\mathrm{B}}\right)^{b_1} \quad \text{…(2.7)}$$

$$\mathrm{R} = \ell n + b_1 + ta_1 \quad \text{… (2.8)}$$

$$\sum_{b_1,b_2,a_1,a_2} = \sum_{a_1=0}^{n}\sum_{a_2=0}^{a_1}\sum_{b_1=0}^{n}\sum_{b_2=0}^{b_1} \quad \text{…(2.9)}$$

E. Generalized Associated Legendre Polynomials

Kuipers and Meulenbeld [10] introduced generalized associated Legendre functions $P_k^{m,n}(z), Q_k^{m,n}(z)$ of first and second kind respectively.

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The function $Q_k^{m,n}(z)$ can be presented in terms of hypergeometric function ${}_2F_1(a,b;c;z)$ as

$$Q_k^{m,n}(z) = e^{m\pi i} 2^{k-\frac{m-n}{2}} \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)}{\Gamma(2k+2)} (z-1)^{-k-\frac{n}{2}-1}(z+1)^{\frac{n}{2}} {}_2F_1\left(k-\frac{m-n}{2}+1, k+\frac{m+n}{2}+1; 2k+2; \frac{2}{1-z}\right) \quad \text{... (2.10)}$$

III. EXPERIMENT AND RESULT

A. First Integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}}{(x-a-1)^{\frac{n}{2}}} S_{V_1,\dots,V_r}^{U_1,\dots,U_r}\left[y_1 \frac{(x-a)^{\sigma_1}(b-x)^{\eta_1}}{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}},\dots, y_r \frac{(x-a)^{\sigma_r}(b-x)^{\eta_r}}{(cx+d)^{\lambda_r}(gx+f)^{\mu_r}}\right]$$

$$\bar{H}_{P,Q}^{M,N}\left[z \frac{(x-a)^{u}(b-x)^{v}}{(cx+d)^{p}(gx+f)^{q}}\right] Q_k^{m,n}\left(1-\frac{2}{x-a}\right) dx$$

$$= (b-a)^{\lambda+\mu+k}(ac+d)^{\gamma}(bg+f)^{\delta} e^{m\pi i} 2^{-\left(\frac{m-n}{2}+1\right)} (-1)^{-\left(k+\frac{n}{2}+1\right)} \Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\sum_{\ell_3=0}^{\infty} (b-a)^{\ell_3} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3} \Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)\,\ell_3!} \sum_{k_1=0}^{[V_1/U_1]} \dots \sum_{k_r=0}^{[V_r/U_r]} (-V_1)_{U_1 k_1} \dots (-V_r)_{U_r k_r} \; A[V_1,k_1;\dots;V_r,k_r]$$

$$\times \prod_{l=1}^{r} \frac{y_l^{k_l} (b-a)^{(\sigma_l+\eta_l)k_l}}{(k_l!)(ac+d)^{\lambda_l k_l}(bg+f)^{\mu_l k_l}} \sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{\ell_1!\ell_2!} \left(\frac{c(a-b)}{ac+d}\right)^{\ell_1} \left(\frac{g(b-a)}{bg+f}\right)^{\ell_2}$$

$$\bar{H}_{P+4,Q+3}^{M,N+4}\left[z \frac{(b-a)^{u+v}}{(ac+d)^{p}(bg+f)^{q}} \middle| \begin{matrix} L_1,(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},L_2 \end{matrix}\right] \quad \text{… (3.1)}$$

where

$$\begin{aligned} L_1 = {} & (1+\gamma-\ell_1-\Sigma\lambda_l k_l, p;1), (-\lambda-k-\ell_1-\ell_3-\Sigma\sigma_l k_l, u;1), \\ & (1+\delta-\ell_2-\Sigma u_l k_l, q;1), (1-\mu-\ell_2-\Sigma\eta_l k_l, v;1) \end{aligned} \quad \text{… (3.2)}$$

$$\begin{aligned} L_2 = {} & (1+\gamma-\Sigma\lambda_l k_l, p;1), (1+\delta-\Sigma\mu_l k_l, q;1), \\ & (-\lambda-\mu-\ell_3-k-\ell_1-\ell_2-\Sigma(\sigma_l+\eta_l)k_l, u+v\,;1) \end{aligned} \quad \text{… (3.3)}$$

The conditions of validity of Eq (3.3) are

(a) Re $(\lambda,\mu) > 0$

(b) $\min\{u, v, p, q\} \geq 0$ (not all zero simultaneously)

(c) when $\min(\sigma_l,\eta_l) \geq 0 \quad (l = 1,\dots,r)$

$1 + \mathrm{Re}(\lambda+k) + u \min\limits_{1 \le j \le M} \mathrm{Re}\,(b_j/\beta_j) > 0$ … (I)

$\mathrm{Re}(\mu) + v \min\limits_{1 \le j \le M} \mathrm{Re}\,(b_j/\beta_j) > 0$ … (II)

When $\max(\sigma_l,\eta_l) < 0 \; (l = 1,\dots,r)$

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$$1+\mathrm{Re}(\lambda+k)+\sum_{l=1}^{r}\left[\sigma_l\left(\frac{\mathrm{V}_l}{\mathrm{U}_l}\right)\right]+u\min_{1\le j\le \mathrm{M}}\mathrm{Re}\,(b_j/\beta_j)>0 \quad \text{… (III)}$$

$$\mathrm{Re}(\mu)+\sum_{l=1}^{r}\left[\eta_l\left(\frac{\mathrm{V}_l}{\mathrm{U}_l}\right)\right]+v\min_{1\le j\le \mathrm{M}}\mathrm{Re}\,(b_j/\beta_j)>0 \quad \text{… (IV)}$$

When $\sigma_l \ge 0,\ \eta_l < 0$, $(l = 1,...,r)$, (I) and (IV) are satisfied.

When $\eta_l \ge 0,\ \sigma_l < 0$, $(l = 1,...,r)$, (II) and (III) are satisfied.

(d) $\max\left\{\left|\frac{c(b-a)}{ac+d}\right|,\left|\frac{g(b-a)}{bg+f}\right|\right\}<1,\ b\ne a.$

(e) If $k+\frac{m+n}{2}\ne -1,-2,...;k\pm\frac{m-n}{2}\ne 0,\pm1,\pm2,...;\quad 2k+2\ne 0,-1,-2,...\quad |z-1|>2$

PROOF. To evaluate the integral, $\mathrm{S}_{\mathrm{V}_1,...,\mathrm{V}_r}^{\mathrm{U}_1,...,\mathrm{U}_r}[x]$ is replaced by its series representation using equation Eq (2.4), $\overline{\mathrm{H}}$-function is replaced by its Mellin-Barnes contour integral form using Eq (2.2) and $Q_k^{m,n}(z)$ is replaced by its hypergeometric function form using Eq(2.10) in the left hand side of Eq(3.1). Then the powers of (x–a), (b–x), (cx+d) and (gx+f) are collected. In the resulting expression the order of integration and summation is interchanged (which is permissible under the conditions stated with (3.1)). The powers of (cx+d) and (gx+f) are simplified by applying following binomial expansions for $x \in [a,b]$

$$(cx+d)^m=(ac+d)^m\sum_{\ell_1=0}^{\infty}\frac{(-m)_{\ell_1}}{\ell_1!}\left\{\frac{-c(x-a)}{ac+d}\right\}^{\ell_1},|(x-a)c|<|ac+d| \quad \text{… (3.4)}$$

$$(gx+f)^n=(bg+f)^n\sum_{\ell_2=0}^{\infty}\frac{(-n)_{\ell_2}}{\ell_2!}\left\{\frac{g(b-x)}{bg+f}\right\}^{\ell_2},\ |g(b-x)|<|bg+f| \quad \text{… (3.5)}$$

and the innermost integral is solved with the help of the following Eulerian type integral

$$\int_a^b (x-a)^{\lambda-1}(b-x)^{\mu-1}\,\mathrm{d}x=(b-a)^{\lambda+\mu-1}\,\mathrm{B}(\lambda,\mu) \quad \text{… (3.6)}$$

Where $\mathrm{Re}\,(\lambda)>0, \mathrm{Re}\,(\mu)>0,\ b\ne a$ and $\mathrm{B}(\lambda,\mu)$ represents beta function

The beta function is simplified in terms of gamma function and resulting Mellin-Barnes contour integral is interpreted as $\overline{\mathrm{H}}$-function. After little simplification, the right hand side of Eq (3.1) is obtained.

B. Second Integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}(x-a-1)^{\frac{n}{2}}}S_{V_1,...,V_r}^{U_1,...,U_r}\left[y_1\frac{(cx+d)^{\lambda_1}(gx+f)^{\mu_1}}{(x-a)^{\sigma_1}(b-x)^{\eta_1}},...,y_r\frac{(cx+d)^{\lambda_r}(gx+f)^{\mu_r}}{(x-a)^{\sigma_r}(b-x)^{\eta_r}}\right]$$

$$\times\ \bar{H}_{P,Q}^{M,N}\left[z\ \frac{(cx+d)^{p}(gx+f)^{q}}{(x-a)^{u}(b-x)^{v}}\right]Q_k^{m,n}\left(1-\frac{2}{x-a}\right)dx$$

$$=(b-a)^{\lambda+\mu+k}(ac+d)^{\gamma}(bg+f)^{\delta}e^{m\pi i}2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\sum_{\ell_3=0}^{\infty}(b-a)^{\ell_3}\frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)\ell_3!}\sum_{k_1=0}^{[V_1/U_1]}...\sum_{k_r=0}^{[V_r/U_r]}(-V_1)_{U_1k_1}...(-V_r)_{U_rk_r}\ A\,[V_1,k_1;...;V_r,k_r]$$

$$\times \prod_{l=1}^{r} \frac{y_l^{k_l}(b-a)^{(\sigma_l+\eta_l)k_l}}{(k_l!)(ac+d)^{\lambda_l k_l}(bg+f)^{\mu_l k_l}} \sum_{\ell_1,\ell_2=0}^{\infty} \frac{1}{\ell_1!\ell_2!}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_1}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_2}$$

$$\times \bar{H}_{P+3,Q+4}^{M+4,N}\left[z\ \frac{(ac+d)^p(bg+f)^q}{(b-a)^{u+v}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},L_1^*\\ L_2^*,(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] \quad \ldots (3.7)$$

where

$$L_1^* = (\gamma-\Sigma\lambda_\ell k_\ell, p), (\delta-\Sigma\mu_\ell k_\ell, q), (1+\lambda+\mu+k+\ell_1+\ell_2+\ell_3-\Sigma(\sigma_\ell+\eta_\ell)k_\ell, u+v) \quad \ldots (3.8)$$

$$L_2^* = (\gamma+\ell_1-\Sigma\lambda_\ell k_\ell, p), (\delta+\ell_2-\Sigma\mu_\ell k_\ell, q), (1+\lambda+\ell_1+\ell_3+k-\Sigma\sigma_\ell k_\ell, u), \quad (\mu+\ell_2-\Sigma\eta_\ell k_\ell, v) \quad \ldots (3.9)$$

The conditions of validity of Eq (3.7) are

(a) $\operatorname{Re}(\lambda,\mu) > 0$

(b) $\min\{u,v,p,q\} \geq 0$ (not all zero simultaneously)

(c) When $\max(\sigma_\ell,\eta_\ell) < 0$ ($\ell = 1,\ldots,r$)

$$1+\operatorname{Re}(\lambda+k) - u \max_{1\le j\le N} \operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right) > 0 \quad \ldots (V)$$

$$\operatorname{Re}(\mu) - v \max_{1\le j\le N} \operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right) > 0 \quad \ldots (VI)$$

When $\min(\sigma_\ell,\eta_\ell) \geq 0$ ($\ell = 1,\ldots,r$)

$$1+\operatorname{Re}(\lambda+k) - \sum_{\ell=1}^{r}\left[\sigma_\ell\left(\frac{V_\ell}{U_\ell}\right)\right] + u \max_{1\le j\le N} \operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right) > 0 \quad \ldots (VII)$$

$$\operatorname{Re}(\mu) - \sum_{\ell=1}^{r}\left[\eta_\ell\left(\frac{V_\ell}{U_\ell}\right)\right] + v \max_{1\le j\le N} \operatorname{Re}\left(\frac{a_j-1}{\alpha_j}\right) > 0 \quad \ldots (VIII)$$

When $\sigma_\ell \geq 0$, $\eta_\ell < 0$ ($\ell = 1,\ldots,r$), (V) and (VII) are satisfied.

When $\eta_\ell \geq 0$, $\sigma_\ell < 0$ ($\ell = 1,\ldots,r$), (VI) and (VIII) are satisfied.

(d) $\max\left\{\left|\frac{(b-a)c}{ac+d}\right|, \left|\frac{(b-a)g}{bg+f}\right|\right\} < 1;\ b \neq a.$

(e) If $k+\frac{m+n}{2} \neq -1,-2,\ldots;\ k \pm \frac{m-n}{2} \neq 0,\pm1,\pm2,\ldots;\quad 2k+2 \neq 0,-1,-2,\ldots$, $|z-1| > 2$

Proof: The integral (3.7) can be evaluated in a similar way as that of the first integral.

C. Third Integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^\gamma(gx+f)^\delta}{(x-a-1)^{\frac{n}{2}}} S_n^{\alpha,\beta,\tau}\left[y\ \frac{(x-a)^\sigma(b-x)^\eta}{(cx+d)^\zeta(gx+f)^\upsilon}\right]$$

$$\times \bar{H}_{P,Q}^{M,N}\left[z\ \frac{(x-a)^u(b-x)^v}{(cx+d)^p(gx+f)^q}\right] Q_k^{m,n}\left(1-\frac{2}{x-a}\right)dx$$

$$= (b-a)^{\lambda+k+\mu}(ac+d)^\gamma(bg+f)^\delta e^{m\pi i}\, 2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\times \sum_{b_1,b_2,a_1,a_2} \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)} \frac{\tau^{\ell_1}y^{R+\ell_1 t}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}$$

$$\frac{(b-a)^{\ell_3+(R+t\ell_1)(\sigma+\eta)}\theta(b_1,b_2,a_1,a_2)}{(ac+d)^{(R+t\ell_1)\zeta}(bg+f)^{(R+t\ell_1)\upsilon}}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_2}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_4} \bar{H}_{P+4,Q+3}^{M,N+4}\left[z\ \frac{(b-a)^{u+v}}{(ac+d)^p(bg+f)^q}\left|\begin{matrix}L_3,(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},L_4\end{matrix}\right.\right] \quad \ldots (3.10)$$

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where

$$L_3=(1+\gamma-\ell_2-(R+t\ell_1)\zeta,p;1),(1+\delta-\ell_4-(R+t\ell_1)\upsilon,q;1),\\(-\lambda-k-\ell_3-\ell_2-(R+t\ell_1)\sigma,u;1),(1-\mu-\ell_4-(R+t\ell_1)\eta,v;1) \qquad \ldots (3.11)$$

$$L_4=\left(1+\gamma-(R+t\ell_1)\zeta,p;1),(1+\delta-(R+t\ell_1)\upsilon,q;1\right),\\\left(-\lambda-\mu-k-(\ell_2+\ell_3+\ell_4)-(R+t\ell_1)(\sigma+\eta),u+v;1\right) \qquad \ldots (3.12)$$

The conditions of validity of Eq (3.10) are

(a) $\mathrm{Re}\,(\lambda,\mu)>0$

(b) $\min\{\sigma,\eta,\zeta,\upsilon,u,v,p,q\}\geq 0$ (not all zero simultaneously)

(c) $1+\mathrm{Re}(\lambda+k)+\mathrm{Re}(\sigma)+u\min\limits_{1\leq j\leq M}\mathrm{Re}\,(b_j/\beta_j)>0$

$\mathrm{Re}(\mu)+\mathrm{Re}(\eta)+v\min\limits_{1\leq j\leq M}\mathrm{Re}\,(b_j/\beta_j)>0$

(d) $\max\left\{\left|\frac{c(b-a)}{ac+d}\right|,\left|\frac{g(b-a)}{bg+f}\right|\right\}<1,\ b\neq a.$

(e) If $k+\frac{m+n}{2}\neq-1,-2,\ldots;k\pm\frac{m-n}{2}\neq 0,\pm1,\pm2,\ldots;\quad 2k+2\neq 0,-1,-2,\ldots$

$|z-1|>2$,

PROOF. To evaluate the integral, the generalized polynomial set $S_n^{\alpha,\beta,\tau}[z]$ is replaced by its series representation form using Eq(2.6), $\overline{H}$ -function is replaced by its Mellin-Barnes contour integral form using Eq(2.2)and $Q_k^{m,n}(z)$ is replaced by its hypergeometric function form using Eq(2.10)in the left hand side of Eq(3.10). Then the powers of (*x*–*a*), (*b*–*x*), (*cx*+*d*) and (*gx*+*f*) are collected. In the resulting expression the order of integration and summation is interchanged (which is permissible under the conditions stated with (3.3)). The integral is simplified using Eqs (3.4) , (3.5) and (3.6) .

The beta function is simplified in terms of gamma function and resulting Mellin-Barnes contour integral is interpreted as $\overline{H}$ -function. After little simplification the right hand side of Eq(3.10) is obtained.

D. Fourth Integral

$$\int_a^b\frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}(x-a-1)^{\frac{n}{2}}}S_n^{\alpha,\beta,\tau}\left[y\;\frac{(cx+d)^{\zeta}(gx+f)^{\upsilon}}{(x-a)^{\sigma}(b-x)^{\eta}}\right]\\\times\overline{H}_{P,Q}^{M,N}\left[z\frac{(cx+d)^p(gx+f)^q}{(x-a)^u(b-x)^v}\right]Q_k^{m,n}\left(1-\frac{2}{x-a}\right)dx$$

$$=(b-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(bg+f)^{\delta}e^{m\pi i}2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\sum_{b_1,b_2,a_1,a_2}\sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty}\frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)}\frac{\tau^{\ell_1}y^{R+\ell_1t}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}$$

$$\times\frac{(ac+d)^{(R+t\ell_1)\zeta}(bg+f)^{(R+t\ell_1)\upsilon}\theta(b_1,b_2,a_1,a_2)}{(b-a)^{(R+t\ell_1)(\sigma+\eta)-\ell_3}}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_2}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_4}$$

$$\times \bar{H}^{M+4,N}_{P+3,Q+4}\left[z \; \frac{(ac+d)^p (bg+f)^q}{(b-a)^{u+v}} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},L_3^* \\ L_4^*,(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] \quad \text{... (3.13)}$$

where

$$L_3^* = \left(\gamma-(R+t\ell_1)\zeta, p\right), \left(\delta-(R+t\ell_1)\upsilon, q\right), \qquad \left(1+\lambda+\mu+k+\ell_2+\ell_3+\ell_4-(R+t\ell_1)(\sigma+\eta), u+v\right) \quad \text{... (3.14)}$$

$$L_4^* = (\gamma+\ell_2-(R+t\ell_1)\zeta, p), (\delta+\ell_4-(R+t\ell_1)\upsilon, q), (1+\lambda+k+\ell_2+\ell_3-(R+t\ell_1)\sigma, u), (\mu+\ell_4-(R+t\ell_1)\eta, v), \quad \text{... (3.15)}$$

and $\theta(b_1,b_2,a_1,a_2), R$ and $\sum_{b_1,b_2,a_1,a_2}$ are as given in Eqs (2.7), (2.8) and (2.9) respectively.

The conditions of validity for (3.13) are

(a) $\mathrm{Re}\,(\lambda,\mu) > 0$

(b) $\min\{\sigma,\eta,\zeta,\nu,u,v,p,q\} \geq 0$ (not all zero simultaneously)

(c) $1+\mathrm{Re}(\lambda+k)-\mathrm{R}\sigma-u \max\limits_{1\le j\le N} \mathrm{Re}\left(\dfrac{a_j-1}{\alpha_j}\right) > 0$

$\mathrm{Re}(\mu)-\mathrm{Re}(\eta)-v \max\limits_{1\le j\le N} \mathrm{Re}\left(\dfrac{a_j-1}{\alpha_j}\right) > 0$

(d) $\max\left\{\left|\dfrac{c(b-a)}{ac+d}\right|, \left|\dfrac{g(b-a)}{bg+f}\right|\right\} < 1,\; b \neq a.$

(e) If $k+\dfrac{m+n}{2} \neq -1,-2,...; k\pm\dfrac{m-n}{2} \neq 0,\pm1,\pm2,...;\quad 2k+2\neq 0,-1,-2,...$

$|z-1| > 2$,

PROOF. The integral (3.13) can be evaluated in a similar way as that of the third integral.

E. Applications

The results obtained from these integrals can be applied to obtain Riemann-Liouville fractional calculus operator of unified functions. Some of the examples are shown below.

(i) Taking $b = z$, $\eta = v = 0$ in Eq(3.1), the Riemann-Liouville fractional calculus operator of order $-\mu$ of a unified function is obtained as

$$ {}_a\mathrm{D}_z^{-\mu}\left\{\frac{(z-a)^{\lambda-1}(cz+d)^{\gamma}(g+zf)^{\delta}}{(z-a-1)^{\frac{n}{2}}} \mathrm{S}^{\mathrm{U}_1,...,\mathrm{U}_r}_{\mathrm{V}_1,...,\mathrm{V}_r}\left[y_1 \frac{(z-a)^{\sigma_1}}{(cz+d)^{\lambda_1}(gz+f)^{\mu_1}},..., \right.\right.$$

$$\left.\left. y_r \frac{(z-a)^{\sigma_r}}{(cz+d)^{\lambda_r}(gz+f)^{\mu_r}}\right] \bar{\mathrm{H}}^{\mathrm{M,N}}_{\mathrm{P,Q}}\left[\frac{z^*(z-a)^u}{(cz+d)^p(gz+f)^q}\right] Q_k^{m,n}\left(1-\frac{2}{z-a}\right)\right\}$$

$$= e^{m\pi i}\, 2^{-\left(\frac{m-n}{2}+1\right)} (-1)^{-\left(k+\frac{n}{2}+1\right)} \Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(\mu) \sum_{\ell_3=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3} \Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)\,\ell_3!}$$

$$(z-a)^{\lambda+\mu+\ell_3+k}(gz+f)^{\delta} \sum_{k_1=0}^{[\mathrm{V}_1/\mathrm{U}_1]} (-\mathrm{V}_1)_{\mathrm{U}_1k_1}...(-\mathrm{V}_r)_{\mathrm{U}_rk_r} \mathrm{A}[\mathrm{V}_1,k_1;...;\mathrm{V}_r,k_r]$$

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$$\prod_{l=1}^{r}\left\{\frac{y_l^{k_l}(z-a)^{\sigma_l k_l}}{k_l!(ac+d)^{\lambda_l k_l}(gz+f)^{\mu_l k_l}}\right\}\sum_{\ell_1,\ell_2=0}^{\infty}\frac{(-1)^{\ell_1}(\mu)_{\ell_2}}{\ell_1!\ell_2!}\left(\frac{c(z-a)}{ac+d}\right)^{\ell_1}\left(\frac{g(z-d)}{gz+f}\right)^{\ell_2}$$

$$\times\overline{\mathrm{H}}_{\mathrm{P+3,Q+3}}^{\mathrm{M,N+3}}\left[\frac{z^*(z-a)^u}{(ac+d)^p(gz+f)^{\mathrm{q}}}\middle|\begin{matrix}\mathrm{L}_1',(a_j,\alpha_j;\mathrm{A}_j)_{1,\mathrm{N}},(a_j,\alpha_j)_{\mathrm{N+1,P}}\\(b_j,\beta_j)_{1,\mathrm{M}},(b_j,\beta_j;\mathrm{B}_j)_{\mathrm{M+1,Q}},\mathrm{L}_2'\end{matrix}\right] \quad \text{.... (5.1)}$$

where

$$\mathrm{L}_1' = (1+\gamma-\ell_1-\Sigma\lambda_l k_l, p;1),\ (-\lambda-k-\ell_1-\ell_3-\Sigma\sigma_l k_l, u;1),\ (1+\delta-\ell_2-\Sigma\mu_l k_l, q;1)$$

$$\mathrm{L}_2' = (1+\delta-\Sigma\mu_l k_l, q;1),\ (1+\gamma-\Sigma\lambda_l k_l, p;1),(1-\lambda-\mu-k-\ell_1-\ell_2-\ell_3-\Sigma\sigma_l k_l, u;1)$$

(ii) Taking $b = z$, $\eta = \nu = 0$ in Eq (3.10), the Riemann-Liouville fractional calculus operator of order $-\mu$ of a unified function is obtained as

$${}_a\mathrm{D}_z^{-\mu}\left\{\frac{(z-a)^{\lambda-1}(cz+d)^{\gamma}(gz+f)^{\delta}}{(z-a-1)^{\frac{n}{2}}}\mathrm{S}_n^{\alpha,\beta,\tau}\left[y\ \frac{(z-a)^{\sigma}}{(cz+d)^{\zeta}(gz+1)^{\upsilon}}\right]\right.$$

$$\left.\overline{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[z^*\ \frac{(z-a)^u}{(cz+d)^p(gz+f)^q}\right]Q_k^{m,n}\left(1-\frac{2}{z-a}\right)\right\}$$

$$= e^{m\pi i}2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(\mu)\ (z-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(gz+f)^{\delta}$$

$$\sum_{b_1,b_2,a_1,a_2}\ \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty}\frac{\theta(b_1,b_2,a_1,a_2)\,y^{\mathrm{R}+t\ell_1}\tau^{\ell_1}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}\frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)}$$

$$\frac{(-1)^{\ell_2}(z-a)^{\ell_3+(\mathrm{R}+t\ell_1)\sigma}(\mu)_{\ell_4}}{(ac+d)^{(\mathrm{R}+t\ell_1)\zeta}(gz+f)^{(\mathrm{R}+t\ell_1)\upsilon}}\left(\frac{c(z-a)}{ac+d}\right)^{\ell_2}\left(\frac{g(z-a)}{gz+f}\right)^{\ell_4}$$

$$.\overline{\mathrm{H}}_{\mathrm{P+3,Q+3}}^{\mathrm{M,N+3}}\left[\frac{z^*(z-a)^u}{(ac+d)^p(gz+f)^q}\middle|\begin{matrix}\mathrm{L}_3',(a_j,\alpha_j;\mathrm{A}_j)_{1,\mathrm{N}},(a_j,\alpha_j)_{\mathrm{N+1,P}}\\(b_j,\beta_j)_{1,\mathrm{M}},(b_j,\beta_j;\mathrm{B}_j)_{\mathrm{M+1,Q}},\mathrm{L}_4'\end{matrix}\right] \quad \text{... (5.2)}$$

where

$$\mathrm{L}_3' = \left(1+\gamma-\ell_2-(\mathrm{R}+t\ell_1)\zeta, p;1\right),\ \left(1+\delta-\ell_4-(\mathrm{R}+t\ell_1)\upsilon, q;1\right),\left(-\lambda-k-\ell_2-\ell_3-(\mathrm{R}+t\ell_1)\sigma, u;1\right)$$

$$\mathrm{L}_4' = \left(1+\gamma-(\mathrm{R}+t\ell_1)\zeta, p;1\right),\ (1+\delta-(\mathrm{R}+t\ell_1)\upsilon, q;1),\left(-\lambda-\mu-k-\ell_2-\ell_3-\ell_4-(\mathrm{R}+t\ell_1)\sigma, u;1\right)$$

and other symbols $\theta(b_1,b_2,a_1,a_2)$, R and $\sum\limits_{b_1,b_2,a_1,a_2}$ are same as given in Eqs(2.7), (2.8) and (2.9) respectively.

The conditions of validity of Eqs(5.1) and (5.2) can be obtained from those stated with (3.1) and (3.10).

Riemann-Liouville fractional calculus operator for the results given by (3.7) and (3.13) can also be obtained in similar manner as discussed above.

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IV.CONCLUSION

In this paper we establish four unified finite integrals involving the product of a general class of a multivariable polynomials, $\bar{H}$ function, generalized associated Legendre function of second kind and generalized polynomial set. A number of several other integrals can also be obtained as special cases of our main results. Since functions and polynomials occurring in these integrals are general in nature, these results can be extended further to provide interesting unifications and extensions of a large number of new and known results.

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