Estimating coefficient bounds with respect to a generalized starlike functions on symmetric points

R.Ambrose Prabhu
Department of Mathematics, Saveetha School of Engineering
Saveetha University, Thandalam, Chennai
Tamilnadu, India

M.Elumalai
Department of Mathematics, Saveetha School of Engineering
Saveetha University, Thandalam, Chennai
Tamilnadu, India

S. Chinthamani
Department of Mathematics, Saveetha School of Engineering
Saveetha University, Thandalam, Chennai
Tamilnadu, India

Abstract - The purpose of the present paper is to estimate the Certain Coefficient for generalized Starlike functions with respect to symmetric points defined on the open unit disk for which $R'_{\phi}(\theta)$ of normalized analytic functions $f(z)$ lies in a region with respect to 1 and symmetric with respect to the real axis.

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1. INTRODUCTION

Let $A$ denote the class of all analytic function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and satisfy the condition $f(0) = 0, f'(0) = 1$. We also denote by $S$ the subclass of $A$ consisting of all functions which are univalent in $\mathbb{D}$. For functions $f(z)$ and $g(z)$ analytic in $\mathbb{D}$, we say that the functions $f(z)$ is said to subordinate to $g(z)$ if there exist a schwarz function $\omega(z)$, analytic in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega'(z)| < 1 \ (z \in \mathbb{D})$, such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{D}).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{D}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\phi(z)$ be an analytic function in $\mathbb{D}$ with $\phi(0) = 1, \phi'(0) > 0$ and $\text{Re}(\phi(z)) > 0, z \in \mathbb{D}$ which map the open unit disk $\mathbb{D}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. Then by $S'_{\phi}$ and $C(\phi)$, respectively, we denote the subclasses of the normalized analytic function class $A$, which satisfy the following subordination relations:

$$\frac{zf''(z)}{f'(z)} < \phi(z), \ z \in \mathbb{D}$$

and
\[
1 + \frac{zf'(z)}{f(z)} < \phi(z), \quad z \in \mathbb{U}.
\]

These function were introduced and studied by Ma and Minda [9]. In particular case, when

\[
\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in \mathbb{U}, 0 \leq \alpha < 1,
\]

these function reduce respectively to the well-known classes \( S^*(\alpha), (0 \leq \alpha < 1) \) of starlike functions of order \( \alpha \) in \( \mathbb{U} \) and \( C(\alpha), (0 \leq \alpha < 1) \) of convex functions of order \( \alpha \) in \( \mathbb{U} \). Ma and Minda [9], the Fekete-Szegö inequality for the functions in the class \( C(\phi) \) was derived and in view of the Alexander result relating the function classes \( S^*(\phi) \) and \( C(\phi) \).

For a brief history of the Fekete-Szegö problems for the starlike, convex and other various subclasses of the normalized analytic function in \( A \), we refer the reader to the work done by Srivatsava et al [20] and Ramachandran et al [14], of course the main result shall refer back to Fekete and szegö [2] in the year 1933. After 30 years or so, Keogh and Merkes [4] solved the problem for certain subclasses of univalent functions.

Koepf [6,7], gave excellent results for the class of close-to-convex functions. These articles [2,4,6,7] gave valuable results which have to solve problems for other extended classes.

Recently Shamnugam et al [17] have studied the Fekete-Szegö problem for subclasses of starlike functions with respect to symmetric points.

Motivated essentially by the aforementioned works, we prove the Fekete-Szegö inequality in Theorem 2.1 below for a more general class of normalized analytic functions.

For \( \lambda, \delta \in \mathbb{N}, \quad k \in \mathbb{N}_0 \), the authors Darus [1] introduced the operator \( D_{k,\delta} \) defined by

\[
D_{k,\delta}f(z) = z + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(1-(n-1)\lambda)\Gamma(n)\Gamma(\delta+1)\right] C(\delta, n) a_n z^n
\]

In the present paper, we obtain the Fekete-szegö inequality for the function \( f \in A \) in the class \( R_{k,\delta}^0(\phi) \) defined as follows:

**Definition:** 1.1 Let \( D_{k,\delta}^f : A \to A \) be a linear operator and \( D_{k,\delta}^f \) is analytic in \( f(\mathbb{U}) \). Let

\[
D_{k,\delta}^f f(z) = z + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(1-(n-1)\lambda)\Gamma(n)\Gamma(\delta+1)\right] C(\delta, n) a_n z^n
\]

where

\[
C(\delta, n) = \frac{\Gamma(n + \delta)}{\Gamma(n)\Gamma(\delta + 1)}
\]

when \( \lambda = 1, \delta = 0 \) we get the sálăgean differential operator, \( k = 0 \) or \( \lambda = 0 \) gives Ruscheweyh operator, \( \delta = 0 \) gives Al-oboudi differential operator of order \( k \)

\[
D_{k,\delta}^0 f(z) = f(z), \quad D_{k,\delta}^1 f(z) = zf'(z)
\]

**Definition:** 1.2 Let \( \phi(z) \) be a univalent starlike function with respect to 1 which map the unit disk \( \mathbb{U} \) onto a region in the right half plane which is symmetric with respect to the real axis \( \phi(0) = 1 \) and \( \phi'(0) > 1 \). A function \( f \in A \) is in the class \( R_{k,\delta}^0(\phi) \) if

\[
\frac{(s-t)z[D_{k,\delta}^1 f(sz)] - [D_{k,\delta}^1 f(tz)]}{[D_{k,\delta}^1 f(sz)] - [D_{k,\delta}^1 f(tz)]} < \phi(z), \quad (\lambda, \delta \in \mathbb{N}, k \in \mathbb{N}_0).
\]

In order to prove our main results, we need the following lemma.

**Lemma:** 1.3 [9] If \( p_v(z) = 1 + c_1 z + c_2 z^2 + \ldots \) is an analytic function with positive real part in \( \mathbb{U} \), then

\[
c_2 - vc_1 \leq \begin{cases} 
2, & \text{if} \quad 0 \leq v \leq 1 \\
4v - 2, & \text{if} \quad v \geq 1.
\end{cases}
\]

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when \( v < 0 \) or \( v > 1 \), the equality holds true if and only if \( p_1(z) = \frac{1 + z}{1 - z} \) or one of its rotations. If \( 0 < v < 1 \), then the equality holds true if and only if \( p_1(z) = \frac{1 + z^2}{1 - z^2} \) or one of its rotations. If \( v = 0 \), then the equality holds true if and only if
\[
p_1(z) = \left( \frac{1}{2} + \frac{1}{2} \gamma \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \gamma \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1),
\]
or one of its rotations. If \( v = 1 \), the equality holds if and only if \( p_1 \) is the reciprocal of one of the functions such that the equality holds in the case of \( v = 0 \). Also the above upper bound is sharp, it can be improved as follows when \( 0 < v < 1 \):
\[
c_2 - vc_3^1 + c_3 \leq 2, \quad 0 < v \leq \frac{1}{2}
\]
and
\[
c_2 - vc_3^1 + (1 - v) c_3 \leq 2, \quad \frac{1}{2} < v \leq 1
\]
We also need the following result in our investigation.

**Lemma 1.4** [15] If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( \mathbb{U} \), then
\[
c_2 - vc_3^1 \leq 2 \max \left( \| \mathbb{I} \|; 2v - 1 \right).
\]
The result is sharp for the functions \( p_1(z) \) given by
\[
p_1(z) = \frac{1 + z^2}{1 - z^2}
\]
and
\[
p_1(z) = \frac{1 + z}{1 - z}.
\]

2. **FEKETE-SZEGŐ PROBLEM FOR THE FUNCTION OF THE CLASS** \( R^d_{\alpha, \beta}(\phi) \).

By making use of Lemma 1.4, we prove the Fekete-szegő Problem for the class \( R^d_{\alpha, \beta}(\phi) \).

**Theorem 2.1.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \). If \( f(z) \) given by (1.1) belongs to the class \( R^d_{\alpha, \beta}(\phi) \), then
\[
a_3 - \mu a_3^2 \leq \begin{cases} 
\Lambda, & \mu \leq \sigma_1 \\
\eta, & \sigma_1 \leq \mu \leq \sigma_2 \\
-A, & \mu \geq \sigma_2.
\end{cases}
\]
where
\[
\sigma_1 = \frac{(\delta + 1)(s + t - 2)(1 + \lambda)^2}{2(\delta + 2)(1 + 2\lambda)^4[(s^2 + st + t^2) - 3]} \left[ \frac{2(B_1 \pm B_3)(s + t - 2) - B_3^1(s + t)}{B_3^1} \right],
\]
\[
\sigma_2 = \frac{(\delta + 1)(s + t - 2)(1 + \lambda)^2}{2(\delta + 2)(1 + 2\lambda)^4[(s^2 + st + t^2) - 3]} \left[ \frac{2B_3(s + t - 2) - (s + t)B_3^2}{B_3^2} \right],
\]
\[
\sigma_3 = \frac{(\delta + 1)(s + t - 2)(1 + \lambda)^2}{2(\delta + 2)(1 + 2\lambda)^4[(s^2 + st + t^2) - 3]} \left[ \frac{(2B_3 - B_3^1)(s + t - 2) - B_3^1(s + t)}{B_3^1} \right],
\]
\[
\Lambda = \frac{4(\delta + 1)(\delta + 2)(1 + 2\lambda)(3 - (s^2 + st + t^2))}{B_3^1}
\times \left[ (B_3^2 - B_3^1) - \frac{B_3^1}{2} \left( \frac{s + t}{s + t - 2} + \frac{2\mu(1 + 2\lambda)^4(\delta + 2)((s^2 + st + t^2) - 3)}{(\delta + 1)(1 + \lambda)^4(2s - s^2 - t^2)} \right) \right],
\]
\[
\eta = \frac{2B_3}{(1 + 2\lambda)^4(\delta + 2)(3 - (s^2 + st + t^2))}.
\]

Further, If \( \sigma_1 \leq \mu \leq \sigma_2 \), then
\[
a_3 - \mu a_3^2 + \frac{(\delta + 1)(s + t - 2)(1 + \lambda)^2}{(\delta + 2)(1 + 2\lambda)^4[(s^2 + st + t^2) - 3]B_3^1} \left[ (B_3^2 - B_3^1)(s + t - 2) + \sigma_3 B_3^1 \right] a_3^2 \leq \eta.
\]
If $\sigma_3 \leq \mu \leq \sigma_2$, then
\[
 a_3 - \mu a_2 = \left(\sigma + 1\right) \left(s + t - 2\right) \left(1 + \lambda^2\right) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) B_1 \left(s + t - 2\right) - \sigma_1 B_1^2 \right) a_1^2 \leq \eta.
\]

Where
\[
 \sigma_4 = \left[\left(s + t\right)\left(s + t - 2\right) \left(1 + \lambda^2\right) + \mu \left(\sigma_2 + 1 + \lambda\right)^2 \left(\sigma_1 + 1 + \lambda\right)^2 \right] \left(s + t - 2\right)\left(1 + \lambda^2\right) .
\]

The result is sharp.

**Proof:** If $f \in R_{j,\lambda}^1(\phi)$, then there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$ and $w(z) < 1$ in $U$ such that
\[
 \frac{(s-t)z}{D^i_{j,\lambda}[f(z)] - D^i_{j,\lambda}[f(z)]} = \phi(w(z)).
\]

Define a function $p_1(z)$ by
\[
 p_1(z) = \frac{1}{1+w(z)}.
\]

since $w(z)$ is a Schwarz function, we see that $\text{Re}(p_1(z)) > 0$ and $p_1(0) = 0$. Define a function $p(z)$ by
\[
 p(z) = \frac{(s-t)z}{D^i_{j,\lambda}[f(z)] - D^i_{j,\lambda}[f(z)]} = \phi(w(z)) = 1 + b_1 z + b_2 z^2 + \cdots
\]

From (2.1), we obtain
\[
 a_2 = \frac{b_1}{(1+\lambda^2)(\sigma+1)(2-s-t)}
\]

and
\[
 a_3 = \frac{b_1}{2} \frac{b_1}{(s+t)(s+t-2)(1+\lambda^2) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) \left(\sigma_2 - \sigma_1\right) B_1 \left(s + t - 2\right) - \sigma_1 B_1^2 \right) a_1^2 \leq \eta.
\]

From (2.2) and (2.4), we get
\[
 b_1 = \frac{1}{2} B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2} B_1 \left(c_2 - \frac{1}{2} c_1^2\right) + \frac{1}{4} B_1 c_2^2
\]

Equating the coefficients of $z$ and $z^2$, we obtain
\[
 b_1 = \frac{1}{2} B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2} B_1 \left(c_2 - \frac{1}{2} c_1^2\right) + \frac{1}{4} B_1 c_2^2
\]

From (2.2) and (2.4), we get
\[
 a_2 = \frac{B_1 c_1}{2(1+\lambda^2)(\sigma+1)(2-s-t)}
\]

and
\[
 a_3 = \frac{B_1}{(1+2\lambda^2)(\sigma+1)(\sigma+2)\left[3-(s^2+st+t^2)\right]} \left(c_2 - c_1^2 \left\{1 - \frac{1}{2} B_1 + \frac{1}{2} B_1 \left(s + t - 2\right)\right\}\right).
\]

Therefore we have
\[ a_i - \mu a_i^3 = \frac{B_i}{[3-(s^2+st+t)](1+2\lambda)^3} \left( c_2 - c_1 \left( 1 - \frac{1}{2} \frac{B_i}{B_i} + \frac{1}{2} \frac{B_i}{B_i} \frac{s+t}{s-t} \right) \right) \]

\[ - \frac{\mu B_i}{(\delta+1)^2(1+\lambda)^3} (2-s-t) \]

where

\[ v = \left[ 1 - \frac{1}{2} \frac{B_i}{B_i} \right] \left( \frac{s+t}{s-t-2} \right) - \frac{\mu B_i (\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]}{(\delta+1)(1+\lambda)^3 (2-s-t)^2} \]

If \( \mu \leq \sigma_1 \), then by Lemma 1.3 and Lemma 1.4, we obtain

\[ a_i - \mu a_i^3 \leq \frac{4(\delta+1)(\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]}{B_i^2} \]

\[ \left( B_i - B_i \right) + \frac{B_i^2}{2} \left( \frac{s+t}{s-t-2} + \frac{2\mu(\delta+2)(1+2\lambda)^3 [(s^2+st+t)-3]}{(\delta+1)(1+\lambda)^3 (2-s-t)^2} \right) \]

which is the first part of Theorem 2.1.

Similarly, if \( \mu \geq \sigma_1 \), then by Lemma 1.3 and Lemma 1.4, we obtain

\[ a_i - \mu a_i^3 \leq \frac{4(\delta+1)(\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]}{B_i^2} \]

\[ \left( B_i - B_i \right) + \frac{B_i^2}{2} \left( \frac{s+t}{s-t-2} + \frac{2\mu(\delta+2)(1+2\lambda)^3 [(s^2+st+t)-3]}{(\delta+1)(1+\lambda)^3 (2-s-t)^2} \right) \]

If \( \sigma_1 \leq \mu \leq \sigma_2 \), we see that

\[ a_i - \mu a_i^3 = \frac{2B_i}{2(1+2\lambda)^3 (\delta+1)(\delta+2) [3-(s^2+st+t)]} (c_2 - c_1) \]

\[ \leq \frac{2B_i}{\delta+1(\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]} \]

Further, If \( \sigma_1 \leq \mu \leq \sigma_2 \), then

\[ a_i - \mu a_i^3 + (\mu - \sigma_1) a_i^3 \leq \frac{2B_i}{\delta+1(\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]} \]

Finally, we see that if \( \sigma_2 \leq \mu \leq \sigma_3 \), then

\[ a_i - \mu a_i^3 + (\sigma_2 - \mu) a_i^3 \leq \frac{2B_i}{\delta+1(\delta+2)(1+2\lambda)^3 [3-(s^2+st+t)]} \]

To show that the bounds are sharp, we define functions \( k_n^\mu (n=2,3,...) \) by

\[ \frac{z(D_{1,\delta}^\mu k_n^\mu(z))}{D_{1,\delta}^\mu k_n^\mu(z)} = \phi(z^{-1}), \quad k_n^\mu(0) = 0 = (k_n^\mu(0))^\gamma - 1, \]

and the function \( F_\gamma \) and \( G_\gamma (0 \leq \gamma \leq 1) \) by

\[ \frac{z(D_{1,\delta}^\mu F_\gamma(z))}{D_{1,\delta}^\mu F_\gamma(z)} = \phi\left(\frac{z+\gamma}{1+\gamma z}\right), \quad F_\gamma(0) = 0 = (F_\gamma(0))^\gamma - 1 \]

and

\[ \frac{z(D_{1,\delta}^\mu G_\gamma(z))}{D_{1,\delta}^\mu G_\gamma(z)} = \phi\left(\frac{z+\gamma}{1+\gamma z}\right), \quad G_\gamma(0) = 0 = (G_\gamma(0))^\gamma - 1 \]

Clearly the functions \( k_n^\mu, F_\gamma \) and \( G_\gamma \in R_{1,\delta}(\phi) \). We also write \( K^\phi = K_n^\mu \).

If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then the equality in Theorem 2.1 holds true if and only if \( f \) is \( K^\phi \) or one of its rotations.

When \( \sigma_1 < \mu < \sigma_2 \), then the equality holds true if and only if \( f \) is \( K_n^\mu \) or one of its rotations. If \( \mu = \sigma_1 \), then the
equality holds true if and only if \( f \) is \( F_r \) or one of its rotations. If \( \mu = \sigma_z \), then the equality holds true if and only if \( f \) is \( G_r \) or one of its rotations.

By making use of Lemma 1.4, we can easily obtain the following theorem.

**Theorem 2.2.** Let \( \phi(z) = 1 + B_1 z + B_2 z^2 + \cdots \), where the coefficients \( B_n \) are real with \( B_1 > 0 \) and \( B_2 \geq 0 \). If \( f(z) \) given by (1.1) belongs to \( R_{\lambda,\mu}^p(\phi) \), then

\[
\alpha_1 \leq \frac{4B_1}{(1 + 2\lambda^2)(\delta + 1)(\delta + 2)(3 - (s^2 + st + t^2))} \times \max \left\{ 1, \frac{2B_1}{B_1} + B_1 \left[ \frac{s + t}{s + t - 2} - \frac{2\mu(\delta + 2)(1 + 2\lambda^2)(3 - (s^2 + st + t^2))}{(\delta + 1)(2 - s - t)(1 + \lambda^4)} \right] \right\},
\]

where \( \mu \in \mathbb{C} \).

The result is Sharp.

**Remark 2.3.** The coefficient bounds for \( a_1 \) and \( a_2 \) are special cases of those asserted by Theorem 2.1.

**Remark 2.4.** In its special case when \( \lambda = 1, \delta = 0 \) and \( k = 0 \), we arrive at a known result due to Ma and Minda [9].

**Remark 2.5.** In its special case when \( \lambda = 1, \delta = 0 \) and \( k = 0 \), \( s = 1 \) and \( t = -1 \), we arrive at a known result due to T.N. Shanmugam et al [17].

2. REFERENCE


