Fixed Point Theorem using Monotone generalised β - nonexpansive Mappings In Banach Space

S. P. Pandey[1], Dr. D.P. Shukla[2]
[1]*Research Scholar Department of Mathematics, Model Science College, Rewa (m.p)*
[2]*Head of Department of Mathematics & Computer Sciences, Model Science College, Rewa (m.p)*

Abstract- We introduce some condition on mappings. We consider a new type of monotone nonexpansive mappings in an ordered Banach space X with partial order $\preceq$. In this paper we define new class of mapping which is known as Monotone generalized $\beta -$ nonexpansive mapping through this mapping we obtain existence theorem.

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I. INTRODUCTION

Let H be a real Hilbert space with norm $\|.\|$ and C be a nonempty subset of H. A mapping T: C→H is said to be L-Lipschitz if there exists L ≥ 0 s.t

$\|T(x) - T(y)\| \leq L\|x - y\|$ For all x, y ∈ C

T is said to be contractive if L∈ [0,1) and is called nonexpansive mapping if L=1. We observe every contractive mapping is nonexpansive and every nonexpansive mapping is Lipschitz.

Following Aoyama and Kohsaka [1], a mapping T: C → H is said to be α-nonexpansive for some real number $\alpha < 1$ if

$\|T(x) - T(y)\|^2 \leq \alpha\|T(x) - y\|^2 + \alpha\|T(y) - x\|^2 + (1 - 2\alpha)\|x - y\|^2 \ \ for\ all\ x, y \in D(T)$. Where D(T) is Domain of T.

Definition 1.1([19]) A mapping T:C→H is said to be k-strictly pseudocontractive if there exists k ∈ [0,1) s.t

$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|x - y - \{T(x) - T(y)\}\|^2$

For all x,y ∈ C

We remark that every k-strictly pseudocontractive mapping is Lipschitz and hence the class of k-strictly pseudocontractive mapping includes properly the class of ninexpansive mappings.

An important class of mapping more general than class of k-strictly pseudocontractive mapping in class of pseudocontractive mapping. T is said to be pseudocontractive if

$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|x - y - \{T(x) - T(y)\}\|^2$

For all x,y∈ C

Definition 1.2([16]) A mapping T:K→K is said to satisfy Condition (C) if for all x,y∈ K

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$$

Implies

$$\|T(x) - T(y)\| \leq \|x - y\|$$

The mapping satisfying condition (C) is also known as Suzuki type generalized nonexpansive mapping.

Definition 1.3([17]) Let (X,$\preceq$) be a partially ordered Banach space and T:X→X be a mapping. The mapping T is said to be monotone if for all x,y∈ X. x $\preceq$ y implies T(x) $\preceq$ T(y)

The concept of nonexpansivity of a map T from a convex set C into C plays an important role in the study of W.R. Mann-type iteration in 1953 given by

$$x_{n+1} = \beta_n T x_n + (1 - \beta_n) x_n, \ \ x_1 \in C \tag{1.1}$$

Here, $\{\beta_n\}$ is a real sequence in [0, 1] satisfying some appropriate conditions, which is usually called a control sequence. The mann iteration method has been extensively investigated for approximating fixed points of nonexpansive mappings. In an infinite dimensional Hilbert space. The mann iteration method can provide only weak convergence

Theorem 1.4([1]) Let K be a nonempty closed convex subset of a uniformly convex Banach space X and T:K→K be an α-nonexpansive mapping. Then F(T) is nonempty if and only if there exists x∈K such that $\{T^n(x)\}$ is bounded.

Remark 1.5 It is interesting to note that nonexpansive are continuous on their domains, but Suzuki-type generalized nonexpansive mapping and α-nonexpansive mapping need not be continuous [16].

On the other hand, fixed point theory is partially ordered metric spaces has been initiated by Ran and Reurings [14] For finding application to matrix equation. Nieto and Lopez [10] extended their result for nondecreasing mapping and presented an application to differential equations. Recently Song et al. [17] extended the notion of α-nonexpansive mapping to monotone α-nonexpansive mapping in order Banach spaces and obtained some existence and convergence theorem for the Mann iteration(see also [15] and the reference therein).

Motivated by the work of Suzuki [16], Aoyama and Kohsaka [1], Bin Dehaish and Khamsi [15], Song et al. [17] and others, we obtain existence results in ordered Banach space for a wider class of nonexpansive mapping[12]. Particularly, in section 3, some auxiliary results are presented. In Section 4, we obtain existence theorems in ordered Banach spaces.

Theorem 1.6([16]) Let K be a nonempty closed convex subset of a Banach space X and T:K→K be a mapping satisfying condition(C). Assume also that either of following holds

(i) K is compact

(ii) K is weakly compact, and X has Opial property.

Then T has a fixed point

II. PRELIMINARIES

Let X be an ordered Banach space with the norm $\|.\|$ and partial order $\preceq$.

Definition 2.1([17]) A subset C of real Banach space X is said to be closed convex cone if the following assumption hold.

(i) C is nonempty and C≠ {0};

(ii) ax+by $\in C$ for x,y ∈ C and a,b ∈ R with a,b $\geq 0;$

(iii) If x$\in C$ and –x ∈ C implies x = 0.

A partial order $\preceq$ in X with respect the closed convex cone C is defined as follows:

x$\preceq$y (x$< y$) $\Leftrightarrow$ y-x ∈ C (y-x ∈ $\mathring{C}$) for all x,y ∈ X where$\mathring{C}$ is an interior of C.

A Banach Space X is said to be uniformly convex in every direction (in Short, UCED) if for each $\varepsilon \in (0,2]$ and z$\in X$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon, z)$$

For all x,y ∈ X with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \in \{tz: t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$.$X$ is said to be uniformly convex if X is UCED and inf$\{\delta(\varepsilon, z): \|z\| = 1\} > 0$.The class of uniformly convex spaces is smaller than the class of UCED spaces.

A Banach space X is said to satisfy the opial property [11] if for every weakly convergent sequence x_n weakly converges in X, we have

$$\lim_{n\to\infty} sup \|x_n - x\| < \lim_{n\to\infty} sup \|x_n - y\|$$

For all y in X with y$\neq x$. It is well known that all Hilbert spaces, all finite dimensional Banach spaces and the Banach spaces l^p(1≤ $p < \infty$) satisfying the opial property, while the uniformly convex spaces $L_p[0,2\pi]$(p$\neq 2$) do not have opial property [8].

Defination 2.2 ([13]) Let K be a subset of a normed space X. A mapping T:K→K is said to satisfy condition(I) if there exists a nondecreasing function f:[0,∞) → [0,∞)

Satisfying f(0) = 0 and f(r)> 0 for all r ∈ $(0, \infty)$ such that $\|x - T(x)\| \geq$ f(d(x,F(T))) for all x∈K, where d(x,F(T) denotes the distance of x from F(T).

Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be bounded sequence in X. For each x∈ X, define:

(i) Asymptotic radius of $\{x_n\}$ at x by r(x, $\{x_n\}$) : = lim $\sup_{n\to\infty} \|x_n - x\|$.

(ii) Asymptotic radius of $\{x_n\}$relative to K by r(x, $\{x_n\}$) : = inf{ r(x, $\{x_n\}$): x∈K}.

(iii) Asymptotic radius of $\{x_n\}$relative to K by A(K, $\{x_n\}$) : = { r(x, $\{x_n\} =$ r(K, $\{x_n\}$): x∈K}.

We note that A(K, $\{x_n\}$) is nonempty. Further, if X is uniformly convex, then A(K, $\{x_n\}$) has exactly one point[8]. Througout, we will assume that order intervals are closed and convex subsets of an ordered Banach space (X,$\preceq$). We denotes these as follows:
[a,$\rightarrow$): = {x $\in X$; a$\preceq x$} and ($\leftarrow$,b]: = {x $\in X$; x$\preceq b$} for any a,b$\in$ X.

III. MONOTONE GENERALIZED β −NONEXPANSIVE MAPPINGS

Definition 3.1 Let K be a nonempty subset of an ordered Bancach space (X,$\preceq$). A mapping T: K$\rightarrow$K will called a monotone generalized β −nonexpansive mappings if T is monotone and there exists $\beta \in [0,1)$ s.t

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$$

Implies

$$\|T(x) - T(y)\| \leq \|x - y\| + \beta\|x - y - \{T(x) - T(y)\}\| \text{ for all x,y}\in\text{K with } x \preceq y \quad (3.1)$$

Now we present some basic properties of generalized β −nonexpansive mappings.

Proposition 3.2 Every Monotone mapping satisfying Condition (C) is a monotone generalized β −nonexpansive mappings but the converse is not true.

Proof Put β=0 in equation (3.1) then we get

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$$

Implies

$$\|T(x) - T(y)\| \leq \|x - y\|$$

Which shows that generalized β −nonexpansive mapping reduce to a mapping satisfying Condition (C) .

The following example shows that the reverse implication does not hold.

Example 3.3([12]) Let K= [0,4] be subset of R endowed with usual order. Define T:K$\rightarrow$K by

$$Tx = \begin{cases} 0 & if\ x \neq 4 \\ 2 & if\ x = 4 \end{cases}$$

Then, for x $\in (2, {}^{8}/_{3}]$ and y = 4,

$$\frac{1}{2}\|x - T(x)\| \leq \|x - y\|$$

Implies

$$\|T(x) - T(y)\| = 2 > \|x - y\|$$

And T does not satisfy Condition (C). Again x $\in (2,3]$ and y = 4,

$$\frac{1}{2}\|y - T(y)\| \leq \|x - y\|$$

Implies

$$\|T(x) - T(y)\| > \|x - y\|$$

And T does not satisfy Condition (C). However, T is β −nonexpansive mappings with $\beta \geq \frac{1}{2}$ and a generalized $\beta -$ nonexpansive mappings with $\geq \frac{1}{3}$.

Proposition 3.4 Let K be a nonempty subset of an ordered Bancach space (X,$\preceq$). A mapping T: :K$\rightarrow$K monotone generalized β −nonexpansive mappings with a fixed point y$\in$K with x$\preceq$y. Then T is monotone quasinonexpansive

Proof It may be completed the proof of Proposition 2[16].

Proposition 3.5 Let K be a nonempty subset of an ordered Bancach space (X,$\preceq$). A mapping T: :K$\rightarrow$K monotone generalized β −nonexpansive mappings. Then F(T) is closed. Moreover, if E is strictly convex and K is convex, then F(T) is also convex.

Proof It may be completed the proof of Proposition 4[16].

The following lemmas will be very useful to prove our main results, which are change on the pattern of [16].

Lemma 3.6 Let K be a nonempty subset of an ordered Bancach space (X,$\preceq$). A mapping T: :K$\rightarrow$K be a monotone generalized β −nonexpansive mapping. Then, for each x,y$\in$K with x$\preceq$y.

(i) $\|T(x) - T^2(x)\| \leq \|x - T(x)\|$;

(ii) $Either\ \frac{1}{2}\|x - T(x)\| \leq \|x - y\|$ or $\|T(x) - T^2(x)\| \leq \|T(x) - y\|$;

(iii) $Either\ \|T(x) - T(y)\| \leq \|x - y\| + \beta\|x - y - \{T(x) - T(y)\}\|$ or
$$\|T^2(x) - T(y)\| \leq \|T(x) - y\| + \beta\|T^2(x) - T(y) - (T(x) - y)\|$$

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Proof It may be completed the proof of Proposition [16] in lemma 5.

Lemma 3.7 Let K be a nonempty subset of an ordered Bancach space $(X,\preceq)$. A mapping T: K→K be a generalized β −nonexpansive mapping. Then, for each x,y ∈ K with $x \preceq y$.

$$\|x-T(y)\| \leq \frac{(2+\beta)}{(1-\beta)}\|x-T(x)\| + \frac{(1+\beta)}{(1-\beta)}\|x-y\|.$$

Proof With the help of Lemma (3.6) we have

$$Either \|T(x)-T(y)\| \leq \|x-y\| + \beta\|x-y-\{T(x)-T(y)\}\|$$

or

$$\|T^2(x)-T(y)\| \leq \|T(x)-y\| + \beta\|T^2(x)-T(y)-(T(x)-y)\|$$

In the first case, we have

$$\begin{aligned}\|x-T(y)\| &= \|x-T(x)+T(x)-T(y)\|\\ &\leq \|x-T(x)\| + \|T(x)-T(y)\|\\ &\leq \|x-T(x)\| + \|x-y\| + \beta\|x-y-\{T(x)-T(y)\}\|\\ &\leq \|x-T(x)\| + \|x-y\| + \beta\|x-T(x)\| + \beta\|y-T(y)\|\end{aligned}$$

This implies that

$$\|x-T(y)\| \leq \frac{(1+\beta)}{(1-\beta)}\|x-T(x)\| + \frac{(1+\beta)}{(1-\beta)}\|x-y\|.$$

In other case, we have

$$\begin{aligned}\|x-T(y)\| &= \|x-T(x)+T(x)-T^2(x)+T^2(x)-T(y)\|\\ &\leq \|x-T(x)\| + \|T(x)-T^2(x)\| + \|T^2(x)-T(y)\|\\ &\leq 2\|x-T(x)\| + \|T^2(x)-T(y)\|\\ &\leq 2\|x-T(x)\| + \|T(x)-y\| + \beta\|T^2(x)-T(y)-(Tx-y)\|\\ &\leq 2\|x-T(x)\| + \|T(x)-y\| + \beta\|x-T(x)\| + \beta\|y-T(y)\|\\ &\leq (2+\beta)\|x-T(x)\| + \|T(x)-y\| + \beta\|y-T(y)\|\end{aligned}$$

This implies

$$\|x-T(y)\| \leq \frac{(2+\beta)}{(1-\beta)}\|x-T(x)\| + \frac{(1+\beta)}{(1-\beta)}\|x-y\|.$$

Therefore in both the case we get the desired result.

IV. MAIN RESULT

4.1 Existence Results

In this section, we present existence theorems for Monotone generalized β −nonexpansive mappings.

Theorem 4.1 Let K be a nonempty closed convex subset of a uniformly convex ordered Bancach space $(X,\preceq)$. Let mapping T: K→K be a monotone generalized β −nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{T^n(x)\}$ is a bounded sequence for some x ∈ K, provided $T^n(x) \preceq y$ for some y ∈ K and $x \preceq T(x)$.

Proof Suppose that $\{T^n(x)\}$ is a bounded sequence for some x ∈ K. Since T is monotone and $x \preceq T(x)$. We get $T(x) \preceq T^2(x)$. Continuing in this way, we get

$T(x) \preceq T^2(x) \preceq T^3(x) \preceq T^4(x)$...

Define $x_n = T^n(x)$ for all n ∈ N. Then the asymptotic centre of $\{x_n\}$ with respect to K is $A(K, \{x_n\}) = \{z\}$ such that $x_n \preceq z$ for all n ∈ N, such z is unique. Now we claim that

$\|x_{n+1} - x_{n+2}\| \leq \|x_n - x_{n+1}\|$.

Since $\frac{1}{2}\|x_n - T(x_n)\| = \|x_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, by (3.1)

$$\begin{aligned}\|x_{n+1} - x_{n+2}\| &= \|T(x_n) - T(x_{n+1})\|\\ &\leq \|x_n - x_{n+1}\| + \beta\|x_n - x_{n+1} - \{T(x_n) - T(x_{n+1})\}\|\\ &\leq \|x_n - x_{n+1}\| + \beta\|x_n - x_{n+1} - x_{n+1} + x_{n+2}\|\\ &\leq (1+\beta)\|x_n - x_{n+1}\| + \beta\|x_n - x_{n+2}\|\end{aligned}$$

$$\|x_{n+1} - x_{n+2}\| \leq \frac{(1+\beta)}{(1-\beta)}\|x_n - x_{n+1}\|$$

Then we can say that because $\beta \in [0,1)$

$\|x_{n+1} - x_{n+2}\| \leq \|x_n - x_{n+1}\|$ (4.1)

Now for all $n \in N$ we claim that either

$\|x_n - x_{n+1}\| \leq 2\|x_n - z\|$ or $\|x_{n+1} - x_{n+2}\| \leq 2\|x_{n+1} - z\|$

To prove above result arguing by contradiction, if possible we consider

$2\|x_n - x_{n+1}\| < \|x_n - z\|$ or $2\|x_{n+1} - x_{n+2}\| < \|x_{n+1} - z\|$

By the triangle inequality

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|x_n - z + z - x_{n+1}\| \\ &\leq \|x_n - z\| + \|z - x_{n+1}\| \\ &\leq \|x_n - z\| + \|x_{n+1} - z\| \\ &< \tfrac{1}{2}\|x_n - x_{n+1}\| + \tfrac{1}{2}\|x_{n+1} - x_{n+2}\| = \|x_n - x_{n+1}\|\end{aligned}$$

$\|x_n - x_{n+1}\| < \|x_n - x_{n+1}\|$

This is contradiction so our supposition is wrong correct one is

$\|x_n - x_{n+1}\| \leq 2\|x_n - z\|$ or $\|x_{n+1} - x_{n+2}\| \leq 2\|x_{n+1} - z\|$

In first case $\frac{1}{2}\|x_n - x_{n+1}\| = \frac{1}{2}\|x_n - T(x_n)\| \leq \|x_n - z\|$

Now by lemma (3.1) we have

$$\begin{aligned}\|T(x_n) - T(z)\| &\leq \|x_n - z\| + \beta\|x_n - z - \{T(x_n) - T(z)\}\| \\ &\leq \|x_n - z\| + \beta\|x_n - T(x_n)\|\end{aligned}$$

$\lim_{n\to\infty} sup\|T(x_n) - T(z)\| \leq \lim_{n\to\infty} sup\|x_n - z\| + \lim_{n\to\infty} sup\|x_n - T(x_n)\|$

$\lim_{n\to\infty} sup\|x_n - T(z)\| \leq \lim_{n\to\infty} sup\|x_n - z\|$

This shows that $T(z) = z$. Similarly we can show second case and we deduce $T(z) = z$.

Conversely, $F(T) \neq \emptyset$. So there exists some $w \in F(T)$ and $T^n(w)= w$ for all $n \in N$. Therefore,$\{T^n(w)\}$ is a constant sequence and ,$\{T^n(w)\}$ and bounded. This completes the proof.

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