Non Homogeneous Heat Conduction Problem and its Thermal Stresses due to Internal Heat Generation in a Semi-Infinite Circular Beam

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Abstract- This article is concerned with the determination of temperature, displacement and thermal stresses in a semi-infinite circular beam under unsteady-state temperature field due to internal heat generation within it. Initially the beam is kept at an arbitrary temperature F(r, z). The boundary surfaces at $(r=a)$, $(z=0)$ and $(z=\infty)$ are kept at temperatures $h(z,t)$, $[-(Q_0/\lambda)g(r,t)]$ and $f(r,t)$ respectively. The governing heat conduction equation has been solved by using Fourier cosine transform and Laplace transform. The results are obtained in terms of Bessel's function in the form of infinite series. The results for temperature change and thermal stresses have been computed numerically and illustrated graphically.
Keywords – Semi-infinite circular beam, thermoelastic problem, Integral transform, direct problem, internal heat source

I. INTRODUCTION

Nowadays non-isothermal problems of the theory of elasticity became very important. This is mainly due to their many applications in widely diverse fields. First, the high velocities of modern aircraft give rise to aerodynamic heating, which produces intense thermal stresses, reducing the strength of the aircraft structure. Second, in the nuclear field, the extremely high temperature and temperature gradients originating inside nuclear reactors influence their design and operation [1].

Nowacki [3] has determined the steady-state thermal stresses in a circular plate subjected to an axisymmetric temperature distribution on the upper surface with zero temperature on the lower surface and with the circular edge thermally insulated. Roy Choudhary [7] has derived the normal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. This satisfies the time-dependent heat conduction equation. Ootao et al. [4] have studied the theoretical analysis of a three-dimensional transient thermal stress problem for a non homogeneous hollow circular cylinder due to a moving heat source in the axial direction from the inner and outer surfaces. Parveen and Khobragade [6] have discussed thermal stresses of a thick circular plate due to heat generation. Varghese and Khobragade [8] have derived alternative solution of a transient heat conduction problem in a circular plate with radiation type boundary conditions.

In the present article, we consider a non homogeneous heat conduction problem in a semi-infinite circular beam occupying the space D: $0 \le r \le a,\ 0 \le z \le \infty$ under unsteady-state temperature field due to internal heat generation within it and discuss the temperature change, displacement and thermal stresses. Initially the beam is kept at an arbitrary temperature F(r, z). The boundary surfaces at $(r=a)$, $(z=0)$ and $(z=\infty)$ are kept at temperatures $h(z,t)$, $[-(Q_0/\lambda)g(r,t)]$ and $f(r,t)$ respectively. The governing heat conduction equation has been solved by using Fourier cosine transform and Laplace transform. The results are obtained in terms of Bessel's function in the form of infinite series. The results for temperature change and thermal stresses have been computed numerically and illustrated graphically.

The results presented here will be more useful in engineering problems particularly, in the determination of the state of strain in circular disc constituting foundations of containers for hot liquids or gases, in the foundations of furnaces etc.

II. STATEMENT OF THE PROBLEM

Consider semi-infinite circular beam defined by $0 \le r \le a,\ 0 \le z \le \infty.$ Let the beam be subjected to a transient axisymmetric temperature field on the radius and axial direction of the cylindrical coordinate system. Initially the temperature of the beam is maintained at F(r, z). The second kind condition $[Q_0\, g(r,t)/\lambda]$ is prescribed over the lower surface (z = 0) and on the upper surface (z = ∞), it is maintained at f(r, t). Under these more realistic prescribed conditions, the transient thermal stresses are required to be determined.

The differential equation governing the displacement potential function $\psi(r,z,t)$ as Noda et al. [2] is

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=K\tau \tag{2.1}$$

where K $\rightarrow$ restraint coefficient and temperature change $\tau = T - T_i$, T_i is initial temperature, displacement function ψ is the Goodier's thermoelastic potential.

The temperature of the plate at time t satisfies the heat conduction equation,

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{\Omega(r,z,t)}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t},\ 0\le r\le a,\ 0\le z\le\infty,\ t\ge 0 \tag{2.2}$$

The boundary conditions are

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=0}=\left(\frac{-Q_0}{\lambda}\right)g(r,t) \qquad 0\le r\le a,\quad t\ge 0 \tag{2.3}$$

$$\left[\frac{\partial T(r,z,t)}{\partial z}\right]_{z=\infty}=f(r,t) \qquad 0\le r\le a,\quad t\ge 0 \tag{2.4}$$

$$\left[T(r,z,t)+\frac{\partial T(r,z,t)}{\partial r}\right]_{r=a}=h(z,t),\ 0\le z\le\infty,\ t\ge 0 \tag{2.5}$$

$$T(r,z,t)=F(r,z),\quad at\ t=0,\ 0\le r\le a,\ 0\le z\le\infty. \tag{2.6}$$

where k is thermal diffusivity of the material of the beam. The displacement function in the cylindrical coordinate system are represented by Michell's function M as Noda et al. [2] are

$$u_r=\frac{\partial\psi}{\partial r}-\frac{\partial^2 M}{\partial r\partial z} \tag{2.7}$$

$$u_z=\frac{\partial\psi}{\partial z}+2(1-\nu)\nabla^2 M-\frac{\partial^2 M}{\partial z^2} \tag{2.8}$$

The Michell's function must satisfy

$$\nabla^2\nabla^2 M=0 \tag{2.9}$$

Where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{2.10}$$

The component of the stresses are represented by the thermoelastic displacement potential ψ and Michel's function M as Noda et al. [2] are

$$\sigma_{rr}=2G\left[\frac{\partial^2\psi}{\partial r^2}-K\tau+\frac{\partial}{\partial z}\left(\nu\nabla^2 M-\frac{\partial^2 M}{\partial r^2}\right)\right] \tag{2.11}$$

$$\sigma_{\theta\theta}=2G\left[\frac{1}{r}\frac{\partial\psi}{\partial r}-K\tau+\frac{\partial}{\partial z}\left(\nu\nabla^2 M-\frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{2.12}$$

$$\sigma_{zz}=2G\left[\frac{\partial^2\psi}{\partial z^2}-K\tau+\frac{\partial}{\partial z}\left((2-\nu)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.13}$$

$$\sigma_{rz}=2G\left[\frac{\partial^2\psi}{\partial r\partial z}+\frac{\partial}{\partial r}\left((1-\nu)\nabla^2 M-\frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.14}$$

For traction free surface, stress functions $\sigma_{rz} = \sigma_{zz} = 0$ at $z = 0 \text{ and } z = \infty$ (2.15)
for solid circular beam.
Equations (2.1) – (2.15) constitute the mathematical formulation of the problem under consideration.

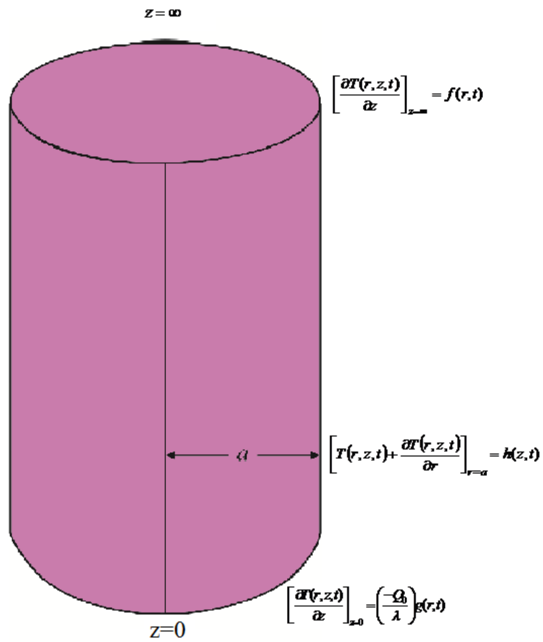

Figure 1 shows the geometry of the problem.

III. SOLUTION OF THE PROBLEM

Applying Fourier cosine transform to the equation (2.1), we get

$$\frac{d^2\overline{T}}{dr^2} + \frac{1}{r}\frac{d\overline{T}}{dr} - p^2\overline{T} = \frac{1}{k}\frac{d\overline{T}}{dt} + \psi$$

Applying Laplace transform to above equation, we get

$$\frac{d^2\overline{T}^*}{dr^2} + \frac{1}{r}\frac{d\overline{T}^*}{dr} - q^2\overline{T}^* = \Gamma^* \qquad (3.1)$$

where $q^2 = p^2 + \frac{s}{k}$

Solution of equation (3.1) is given by

$$\overline{T}^*(r,n) = \left\{\overline{g}(n) - [\psi(a) + \psi'(a)]\right\}\left(\frac{I_0(qr)}{I_0(qa) + J_0'(qa)}\right) + \Gamma \qquad (3.2)$$

where $\Gamma = P.I$.

Applying inverse Laplace transform and Fourier cosine transform to above equation, we get

$$T(r,z,t) = \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\cos(pz)J_0(r\lambda_m) \times \int_0^t \overline{F}\, e^{-k(\lambda_m^2+q^2)(t-t')}dt' + \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}\sin(pz)\Pi \qquad (3.3)$$

where $\Pi = L^{-1}(\Gamma)$, $\overline{F} = -\{\Gamma(a) + \Gamma'(a)\}$,

$$A_{mn} = \frac{4K}{a}\left(\frac{\lambda_m}{3J_1(a\lambda_m) + J_0(a\lambda_m)}\right), \quad q^2 = p^2 + \frac{s}{k}, \quad p^2 = n^2\pi^2$$

λ_m are the positive roots of a transcendental equation $J_0(\lambda_m a) = 0.$
Assume that Michel's function M which satisfies condition (2.9) as

$$M = \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}\cos(pz)\Pi \tag{3.4}$$

Using equations (3.1) and (3.2) in equation (2.1), we obtain the displacement potential function ψ as

$$\psi = \left(\frac{Q_0}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos(pz)J_0(r\lambda_m)B(t) - \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}Kp\sin(pz)\Pi \tag{3.5}$$

Further using equations (3.3), (3.4) and (3.5) in equations (2.7), (2.8) and (2.11) to (2.14), we obtain expression for displacement components and thermal stresses as

$$u_r = \left(\frac{Q_0}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\lambda_m\cos(pz)J_1(r\lambda_m)B(t) - \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}(K-1)p\sin(pz)\Pi' \tag{3.6}$$

$$u_z = -\left(\frac{Q_0}{\lambda}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}p\sin(pz)J_0(r\lambda_m)B(t) - \left(\frac{Q_0}{\lambda}\right)\sum_{n=1}^{\infty}Kp\sin(pz)\Pi' + 2(1-\nu)\left(\frac{Q_0}{\lambda}\right)\left\{\sum_{n=1}^{\infty}\cos(pz)\,\Pi'' + \frac{1}{r}\sum\cos(pz)\,\Pi' - \sum p^2\cos(pz)\,\Pi\right\} \tag{3.7}$$

$$\begin{aligned}\sigma_{rr} = \frac{2GQ_0}{\lambda}\Bigg\{&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_m^2 B_{mn}\cos(pz)J_1'(r\lambda_m)B(t) - \sum_{n=1}^{\infty}Kp\sin(pz)\Pi'' \\ &- K\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos(pz)J_0(r\lambda_m)\int_0^t \overline{F}\, e^{-k(\lambda_m^2+q^2)(t-t')}dt' - \sum_{n=1}^{\infty}p^2\cos(pz)\,\Pi\right\} \\ &- \frac{\nu}{r}\sum_{n=1}^{\infty}p\sin(pz)\,\Pi' + \nu\sum_{n=1}^{\infty}p^3\sin(pz)\,\Pi\Bigg\}\end{aligned} \tag{3.8}$$

$$\begin{aligned}\sigma_{\theta\theta} = \frac{2GQ_0}{\lambda}\Bigg\{&\frac{1}{r}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_m B_{mn}\cos(pz)J_1(r\lambda_m)B(t) - \frac{1}{r}\sum_{n}Kp\sin(pz)\,\Pi \\ &- K\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos(pz)J_0(r\lambda_m)\int_0^t \overline{F}\, e^{-k(\lambda_m^2+q^2)(t-t')}dt' - \sum p^2\cos(pz)\,\Pi\right\} \\ &- \nu\sum_{n=1}^{\infty}p\sin(pz)\,\Pi'' + \nu\sum_{n=1}^{\infty}p^3\sin(pz)\,\Pi\Bigg\}\end{aligned} \tag{3.9}$$

$$\begin{aligned}\sigma_{zz} = \left(\frac{2GQ_0}{\lambda}\right)\Bigg\{&-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}p^2\cos(pz)J_0(r\lambda_m)B(t) + \sum_{n=1}^{\infty}Kp^3\sin(pz)\Pi' \\ &- K\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos(pz)J_0(r\lambda_m)\int_0^t \overline{F}\, e^{-k(\lambda_m^2+q^2)(t-t')}dt' - \sum_{n=1}^{\infty}p^2\cos(pz)\,\Pi\right\}\end{aligned}$$

$$-(2-\nu)\sum_{n=1}^{\infty} p\sin(pz)\,\Pi'' - \frac{1}{r}\sum_{n=1}^{\infty} p\sin(pz)\Pi' + (1-\nu)\sum_{n=1}^{\infty} p^3\sin(pz)\,\Pi\Bigg\} \tag{3.10}$$

$$\sigma_{rz} = \left(\frac{2GQ_0}{\lambda}\right)\Bigg\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_m B_{mn}\cos(pz)J_1(r\lambda_m)B(t) - \sum_{n=1}^{\infty} Kp\sin(pz)\,\Pi'$$

$$+(1-\nu)\left\{\frac{1}{r}\sum_{n=1}^{\infty}\cos(pz)\,\Pi'' + \sum_{n=1}^{\infty} p^3\sin(pz)\,\Pi'''\right\} - \nu\sum_{n=1}^{\infty} p^3\sin(pz)\,\Pi'\Bigg\} \tag{3.11}$$

where

$$A_{mn} = \frac{4K}{a}\left(\frac{\lambda_m}{3J_1(a\lambda_m)+J_0(a\lambda_m)}\right), \quad B_{mn} = K\,A_{mn}, \quad B(t) = \int\left(\int_0^t \overline{F}\, e^{-k(\lambda_m^2+q^2)(t-t')}dt'\right)dt$$

IV. SPECIAL CASE AND NUMERICAL CALCULATIONS

Setting $F(r,z) = \delta(r-r_0)\times e^{-z^2}$ (4.1)

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$\Omega(r,z,t) = \Omega_i\delta(r-r_i)\,\delta(z-z_i)\delta(t-\tau)$

where r is the radius of the beam and δ is the Dirac – delta function.

The heat source $\Omega(r,z,t)$ is an instantaneous line heat source of strength Ω_i = 50 Btu/hr.ft3, situated at center of the beam along the radial direction and axial direction and releases its instantaneously at the time t = 1 hr.

V. DIMENSIONS

The numerical computation have been carried out for Aluminum metal with parameter
radius of a circular beam a =5 ft,
Height a circular beam = 1000 ft,
Time t = 1 hour.

VI. MATERIAL PROPERTIES

The numerical calculation has been carried out for an Aluminum (Pure) circular beam with the material properties [1] as,

Density ρ = 169 lb/ft3
Specific heat cp= 0.208 Btu/ lb0F
Thermal conductivity k = 117 Btu/(hr.ft0F)
Thermal diffusivity α = 3.33 ft2/hr
Poisson ratio ν = 0.35
Coefficient of linear thermal expansion α_t = 12.84 $\times 10^{-6}$ 1/F
Lame constant μ = 26.67
Young's Modulus of elasticity E =70GPa

VII. ROOTS OF THE TRANSCENDENTAL EQUATION

The first six roots of the transcendental equation $J_0(\lambda_m a) = 0$ for a = 5 as defined in Ozisik [5] are $\lambda_1 = 0.4809$, $\lambda_2 = 1.1040$, $\lambda_3 = 1.7307$, $\lambda_4 = 2.3583$, $\lambda_5 = 2.9861$ and $\lambda_6 = 3.6142$.

The numerical calculation was carried out with help of computational mathematical software Mathcad-2007, and the graphs were plotted using Excel (MS Office-2007).

VIII. DISCUSSION

In this study, we analyzed the non-homogeneous heat conduction problem in a semi-infinite circular beam. As an illustration, we carried out numerical calculations for a beam made up of aluminum (Pure). The thermoelastic behavior is examined such as displacement function and thermal stresses with the help of temperature and radial as well as axial displacement.

From Figure 2, it is observed that the temperature increases as the radius of beam increases. From Figure 3, it is observed that the displacement potential function steadily increases as the radius of beam increases. From Figure 4 and 5, it is observed that the displacement component increases and then becomes steady as the radius of beam increases. From Figure 6, 7, 8 and 9 it is observed that the stress function increases as the radius of beam increases.

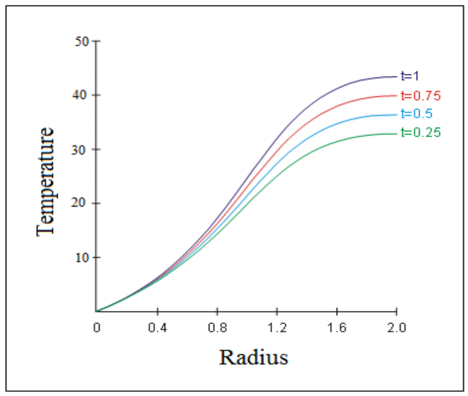

Figure 2 : Variation of temperature versus radius

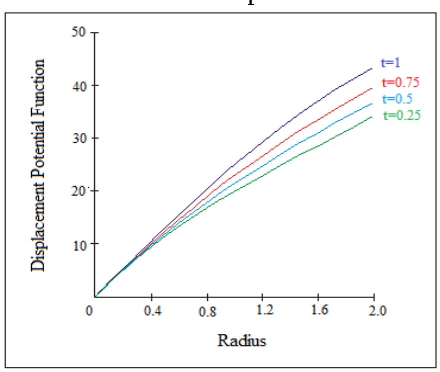

Figure 3 : Variation of displacement potential function versus radius

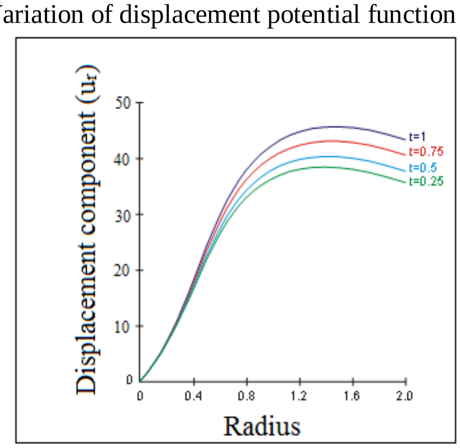

Figure 4 : Variation of displacement component (ur)versus radius

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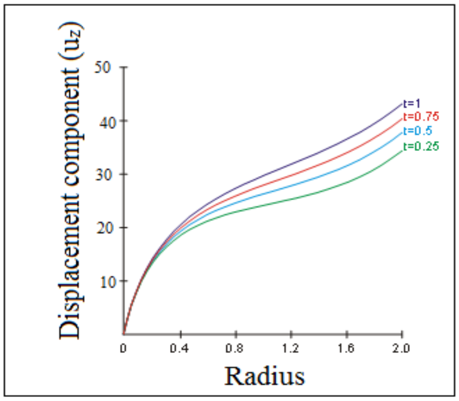

Figure 5 : Variation of displacement component (uz)versus radius

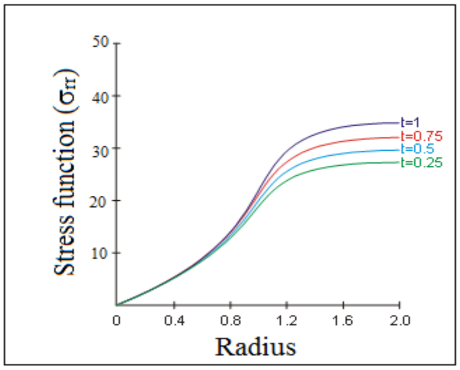

Figure 6 : Variation of Stress function (σrr)versus radius

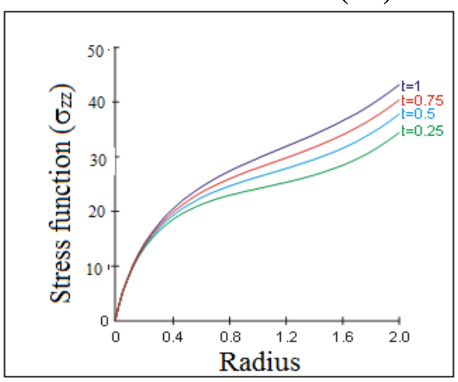

Figure 7 : Variation of Stress function (σzz)versus radius

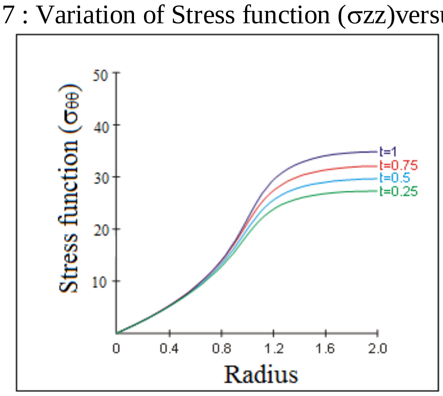

Figure 8 : Variation of Stress function (σθθ)versus radius

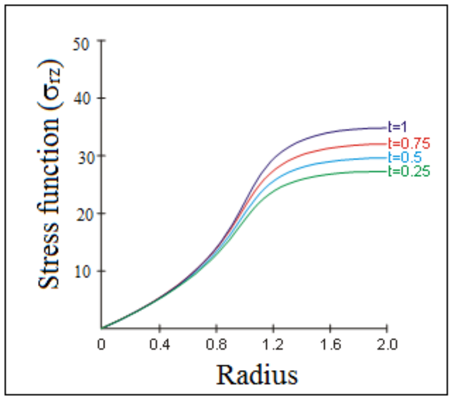

Figure 9 : Variation of Stress function (σrz)versus radius

IX. CONCLUSION

The temperature distribution, displacement and thermal stresses of a non homogeneous heat conduction problem in a semi-infinite circular beam under unsteady-state temperature field due to internal heat generation have been investigated. The present method is based on direct method. Using the Fourier cosine transform and Laplace transform, the numerical results have been calculated for Aluminium metal of a beam. We conclude that, due to internal heat generation in a semi-infinite circular beam, the temperature increases as the radius of beam increases.
The results presented here will be useful in engineering problems, particularly in aerospace engineering for stations of a missile body not influenced by nose tapering. Also, any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions.

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