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# Fuzzy Fixed Point Mappings on Metric Space

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**Abstract:- M.A. Ahamed [1] gave the refined and generalized the common fixed point theorem, which have been proved by S.C. Arora and C. Sharma [2]. In this paper, we shall improve the theorem of M.A. Ahamed [1]. Also, we establish the error estimation as well as the rate of convergence of generalized common fixed point theorem.**
**Key Words:- Fuzzy mappings, Fuzzy fixed point mappings, Fixed point theorem, Metric spaces.**

## I. INTRODUCTION

The concept of Fuzzy sets was investigated by L.A. Zadeh [16] in 1965. Fuzzy metric space was introduced by Kramosil and Michalek [10] in 1975. Then in 1994, the notion of fuzzy metric spaces was modified by George and Veera-mani [6]. Many researchers have been obtained the common fixed point theorems for self mappings with different types of contraction and commu-tativity conditions. Sessa [14] was initiated the weakly commuting maps on metric spaces to improve commutativity in fixed point theorems, later on, this method was enlarged to compatible maps by Jungck [9]. Then Tas et.al [15] was extended the Jungck's compatibility conditions to four self mappings on complete and compact metric spaces. Recently, Ahamed [1] generalized the improved results of S.C. Arora and V. Sharma [2].

This paper widely inspired by Tas et al. [15] and Ahamed [1]. We give different approach of Ahamed's results and we establish the error estimation as well as the rate of convergence of common fuzzy fixed point mappings on metric spaces.

## II. PRELIMINARY NOTES

Let X be any metric space with the metric d and I = [0, 1] be unit in-terval. A fuzzy set A in a metric space X is said to be an approximate quantity if and only if for each $\alpha \in I$ the $\alpha$-level set of A is non empty compact convex set in X and $\sup_{x\in X} A(x) = 1$. W (X) is the family of all approximate quantities in X. That is, for any $\alpha \in I$, W (X) is given by $\{A_\alpha \in I^X : A_\alpha$ is non empty compact convex set with $\sup_{x\in X} A(x) = 1\}$, where $I^X$ is collection of fuzzy subsets of X.

Note that, a set A is more accurate than the set B in W (X), denoted by $A\subset B$, if and only if $A(x) \leq B(x)$ for each $x \in X$, where A(x), B(x) denotes the membership values of x in X. For $x \in X$, we write {x} the characteristic function of the ordinary subset {x} of X. We denote $W_0(X) = \{\{x\} : x \in X\}$.

For some $\alpha \in I$ and $A, B \in W(X)$,

$$p_\alpha(A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y); \qquad D_\alpha(A, B) = H(A_\alpha, B_\alpha);$$

$$p(A, B) = \sup_{\alpha\in I} p_\alpha(A, B); \qquad \text{and} \quad D(A, B) = \sup_{\alpha\in I} D_\alpha(A, B),$$

where H is the Hausdorff metric induced by the metric d, $p_\alpha$ is a non-decreasing function of $\alpha$ and D is a metric on W (X).

Definition 1: [5] Let Y be an arbitrary set, X be a metric linear space. A mapping $T : Y \to W(X)$ is said to be a fuzzy mapping, if for each $y \in Y$, $Ty \in W(X)$. Thus if we characterize a fuzzy set Ty in a metric linear space X by a member ship function Ty, then Ty(x) is the grade of member ship of x in Ty.

Note that, a fuzzy mapping T is a fuzzy subset on X×Y with membership function $T_x(y)$.

Definition 2: [14] Self-mappings f and g on a metric space (X, d) are said to weakly commute if and only if $d(fgx, gfx) < d(fx, gx)\ \forall x \in X$.

Definition 3: [9] Self-mappings f and g on a metric space (X, d) are said to be compatible if and only if whenever $x_n$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, then $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

The following proposition and lemmas are needed in the sequel.

Proposition 1: [9] Let A, B be compatible self mappings on a complete metric space (X, d).

If for some $t \in X$, At = Bt, then ABt = BAt.

Suppose that $\lim_{n\to\infty} Ax_n = t = \lim_{n\to\infty} Bx_n$, for some $t \in X$.

If A is continuous at $t \in X$, then $\lim_{n\to\infty} BAx_n = At$.
If A & B are continuous at $t \in X$, then $At = Bt$ and $ABt = BAt$.
Lemma 1: [1] If $\{x_0\} \subset A$ for each $A \in W^*(X)$ and $x_0 \in X$, then $p_\alpha(x_0, B) \leq D_\alpha(A, B)$ for each $B \in W^*(X)$.
Lemma 2: [1] $p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$, $\forall x, y \in X$ and $A \in W^*(X)$.
Lemma 3: Let (X, d) be a complete metric space and $A, B : X \to W(X)$ be fuzzy mappings. Assume that there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$, such that for all $x, y \in X$,

$$D^2(Ax, By) \leq c_1 \max\{d^2(x, y), p^2(x, Ax), p^2(y, By)\}$$

$$c_2 \max\{p(x, Ax)p(x, By), p(y, Ax)p(y, By)\}$$

$$+ c_3 p(x, By)p(y, Ax). \quad (2.1)$$

Then for some $x_0 \in X$ there exists $x_1 \in X$ such that $\{x_1\} \subset Ax_0$ and the sequence

$$\{y_n\} = \{Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots\} \quad (2.2)$$

is Cauchy.
Lemma 4: Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset X \Longleftrightarrow p_\alpha(x, A) = 0$ for each $\alpha \in I$.

## III. MAIN RESULT

Ahamed [1] proved the following result;
Theorem 1: Let $(X, d)$ be a complete metric space and $T_1, T_2$ be fuzzy map-pings from X into $W^*(X)$. Assume that there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1,$ such that for all $x, y \in X,$

$$D^2(T_1(x), T_2(y)) \leq c_1 \max\{d^2(x, y), p^2(x, T_1(x)), p^2(y, T_2(y))\}$$
$$+ c_2 \max\{p(x, T_1(x))p(x, T_2(y)), p(y, T_1(x))p(y, T_2(y))\}$$

$$+ c_3 p(x, T_2(y))p(y, T_1(x)). \quad (3.1)$$

Then, there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

In the above result, $W^*(X)$ is a sub collection of fuzzy subsets of X. In fact, each element in W(X) leads to in $W^*(X)$ but converse is not true, this implies, $W(X) \subset W^*(X)$. So, we establish the modified result of Ahamed's as follows:

Theorem 2: Let (X, d) be a complete metric space, $A, B : X \to W(X)$ be fuzzy mappings. Suppose there exist $c_1, c_2, c_3 \in I$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$, such that for all $x, y \in X$,

$$D^2(Ax, By) \leq c_1 \max\{d^2(x, y), p^2(x, Ax), p^2(y, By)\}$$
$$+ c_2 \max\{p(x, Ax)p(x, By), p(y, Ax)p(y, By)\}$$

$$+ c_3 p(x, By)p(y, Ax). \quad (3.2)$$

Then,

there exists $z \in X$, such that $\{z\} \subset Az$ and $\{z\} \subset Bz$.

a priori error estimation:

$$d(x_n, x_{n+1}) \leq (c_1 + 2c_2)^{n/2} d(x_0, x_1)$$
and

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$$d(x_n, z) \le \sum_{k=0}^{\infty} (c_1 + 2c_2)^{n+k} d(x_0, x_1).$$

iii) a posteriori error estimation:

$$d(x_n, z) \le \sum_{k=0}^{\infty} (c_1 + 2c_2)^k d(x_n, x_{n+1}).$$

iv) the rate of convergence

$$d(x_n, z) \le (c_1 + 2c_2)^{n/2} d(x_0, z).$$

Proof: Since lemma 3, for any arbitrary point $x_0 \in X$, there exists $x_1 \in X$ such that $\{x_1\} \subset Ax_0$ and for $x_1 \in X$ there exists $x_2 \in X$ such that $\{x_2\} \subset Bx_1$, where $Ax_0$, $Bx_1$ are non-empty compact convex subsets of X. This implies that, for $x_1 \in Ax_0$, there exists $x_2 \in Bx_1$ such that

$$d(x_1, x_2) \le D_1(Ax_0, Bx_1) \le D(Ax_0, Bx_1). \tag{3.3}$$

$$\Rightarrow d^2(x_1, x_2) \le D^2(Ax_0, Bx_1)$$

$$c_1 \max\{d^2(x_0, x_1), p^2(x_0, Ax_0), p^2(x_1, Bx_1)\}$$

$$c_2 \max\{p(x_0, Ax_0)p(x_0, Bx_1), p(x_1, Ax_0)p(x_1, Bx_1)\}$$

$$c_3 p(x_0, Bx_1)p(x_1, Ax_0)$$

$$c_1 \max\{d^2(x_0, x_1), d^2(x_1, x_2)\} + c_2 p(x_0, x_1)p(x_0, x_2)$$

$$c_1 \max\{d^2(x_0, x_1), d^2(x_1, x_2)\}$$

$$c_2\, d(x_0, x_1)[d(x_0, x_1) + d(x_1, x_2)].$$

Suppose $d(x_0, x_1) < d(x_1, x_2)$, then $d^2(x_0, x_1) < d^2(x_1, x_2)$.

$$\Rightarrow d^2(x_1, x_2) \le c_1 d^2(x_1, x_2) + 2c_2\, d(x_0, x_1)d(x_1, x_2)$$

$$(c_1 + 2c_2)d^2(x_1, x_2).$$

This implies that, $c_1 + 2c_2 \ge 1$, which is contradict to $c_1 + 2c_2 < 1$.

$\therefore\ d(x_0, x_1) > d(x_1, x_2)$,

$$\Longrightarrow d^2(x_1, x_2) < d^2(x_0, x_1).$$

$$\Longrightarrow d^2(x_1, x_2) \le c_1 d^2(x_0, x_1) + 2c_2\, d^2(x_0, x_1)$$

$$(c_1 + 2c_2)\, d^2(x_0, x_1).$$

$$\Longrightarrow d(x_1, x_2) \le (c_1 + 2c_2)^{1/2} d(x_0, x_1).$$

Similarly,

$$d(x_2, x_3) \le (c_1 + 2c_2)\, d(x_0, x_1),$$

$$d(x_3, x_4) \le (c_1 + 2c_2)^{3/2}\, d(x_0, x_1),$$

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$$d(x_n, x_{n+1}) \leq (c_1 + 2c_2)^{n/2} \, d(x_0, x_1). \qquad (3.4)$$

Since the lemma 3, for any arbitrary $x_0 \in X$, there exists $x_1, x_2 \in X$ such that $Ax_0 \supset \{x_1\}$ and $Bx_1 \supset \{x_2\}$. In this process we can construct a cauchy sequence as follows:

$Ax_2 \supset \{x_3\}, Bx_3 \supset \{x_4\}, Ax_4 \supset \{x_5\}, Bx_5 \supset \{x_6\}, \ldots$

$\ldots, Ax_{2n} \supset \{x_{2n+1}\}, Bx_{2n+1} \supset \{x_{2n+2}\}, \ldots.$

this implies, $\{x_{2n+1}\} \subset Ax_{2n}$ and $\{x_{2n+2}\} \subset Bx_{2n}$, for $n = 0, 1, 2, \ldots.$

By lemma 2 and the equation 3.4, for each $n = 0, 1, 2, \ldots$ we have,

$$d^2(BAx_0, (BA)^2x_0) \leq (c_1 + 2c_2)^2 d^2(x_0, BAx_0),$$
$$d^2((BA)^2x_0, (BA)^4x_0) \leq (c_1 + 2c_2)^4 d^2(x_0, BAx_0),$$

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$$d^2((BA)^n x_0, (BA)^{n+2}x_0) \leq (c_1 + 2c_2)^{2n} d^2(x_0, BAx_0),$$

$$\Longrightarrow d((BA)^n x_0, (BA)^{n+2}x_0) \leq (c_1 + 2c_2)^n d(x_0, BAx_0)$$

Similarly,

$$d((AB)^n x_1, (AB)^{n+1}x_1) \leq (c_1 + 2c_2)^n d(x_1, ABx_1).$$

Now, for some $m, n \in N$ consider

$$d((BA)^n x_0, (BA)^{n+m}x_0) \leq d((BA)^n x_0, (BA)^{n+2}x_0) + d((BA)^{n+2}x_0, (BA)^{n+4}x_0)$$

$$+ \ldots + d((BA)^{n+m-2}x_0, (BA)^{n+m}x_0)$$

$$\leq (c_1 + 2c_2)^n d(x_0, BAx_0) + (c_1 + 2c_2)^{n+2} d(x_0, BAx_0)$$

$$+ \ldots + (c_1 + 2c_2)^{n+m-2} d(x_0, BAx_0)$$

$$\leq [(c_1 + 2c_2)^n + (c_1 + 2c_2)^{n+1} + (c_1 + 2c_2)^{n+2}$$

$$+ \ldots + (c_1 + 2c_2)^{n+m-1}] d(x_0, BAx_0)$$

$$\leq (c_1 + 2c_2)^n [1 + (c_1 + 2c_2) + (c_1 + 2c_2)^2 + (c_1 + 2c_2)^3$$

$$+ \ldots + (c_1 + 2c_2)^{m-1}] d(x_0, BAx_0)$$

$$\leq \frac{(c_1 + 2c_2)^n}{1 - c_1 - 2c_2} d(x_0, BAx_0).$$

Since $c_1 + 2c_2 < 1$ and the metric $d$ is continuous,

$$\Longrightarrow \lim d((BA)^n x_0, (BA)^{n+m}x_0) = 0.$$

$m,n\to\infty$

Similarly, for some $m, n \in N$

$d((AB)^n x_0, (AB)^{n+m} x_0) \leq d((AB)^n x_0, (AB)^{n+2} x_0) + d((AB)^{n+2} x_0, (AB)^{n+4} x_0)$

$+ \ldots + d((AB)^{n+m-2} x_0, (AB)^{n+m} x_0)$
$\leq (c_1 + 2c_2)^n d(x_0, ABx_0) + (c_1 + 2c_2)^{n+2} d(x_0, ABx_0)$
$+ \ldots + (c_1 + 2c_2)^{n+m-2} d(x_0, ABx_0)$
$\leq [(c_1 + 2c_2)^n + (c_1 + 2c_2)^{n+1} + (c_1 + 2c_2)^{n+2}$
$+ \ldots + (c_1 + 2c_2)^{n+m-1}] d(x_0, ABx_0)$
$\leq (c_1 + 2c_2)^n [1 + (c_1 + 2c_2) + (c_1 + 2c_2)^2 + (c_1 + 2c_2)^3$
$+ \ldots + (c_1 + 2c_2)^{m-1}] d(x_0, ABx_0)$
$\leq \frac{(c_1 + 2c_2)^n}{1 - c_1 - 2c_2} d(x_0, ABx_0).$

Since $c_1 + 2c_2 < 1$ and the metric d is continuous, this implies that,

$\lim_{m,n\to\infty} d((AB)^n x_0, (AB)^{n+m} x_0) = 0,$

and

$\lim_{n\to\infty} d((BA)^n x_0, (AB)^n x_1) \leq \lim_{n\to\infty} d(x_{2n}, x_{2n+1}) = \lim_{n\to\infty} (c_1 + 2c_2)^n d(x_0, x_1) = 0.$

Hence by lemma 3, the sequences $(AB)^n$ and $(BA)^n$ are converges uniformly in X. Therefore there exists $N \in N$ and $z \in X$, such that

$$\lim_{n\to\infty} (AB)^n x_1 = z = \lim_{n\to\infty} (BA)^n x_0, \quad \forall n \geq N. \tag{3.5}$$

That is, the mappings AB & BA are compatible and there exists $z \in X$, such that $\{z\} \subset ABz$ and $\{z\} \subset BAz$. This implies that, for each $\alpha \in I$

$p_\alpha(z, Az) \leq d(z, x_{2n+1}) + p_\alpha(x_{2n+1}, Az) \leq d(z, x_{2n+1}) + D_\alpha(Ax_{2n}, Az).$

$$p(z, Az) \leq d(z, x_{2n+1}) + p(x_{2n+1}, Az)$$
$$\leq d(z, x_{2n+1}) + D(Ax_{2n}, Az). \tag{3.6}$$

From inequality (3.7),

$$\begin{aligned}
D^2(Ax_{2n}, Az) &\leq c_1 \max\{d^2(x_{2n}, z), p^2(x_{2n}, Ax_{2n}), p^2(z, Az)\} \\
&\quad + c_2 \max\{p(x_{2n}, Ax_{2n})p(x_{2n}, Az), p(z, Ax_{2n})p(z, Az)\} \\
&\quad + c_3 p(x_{2n}, Az)p(z, Ax_{2n}) \\
&\leq c_1 \max\{d^2(x_{2n}, z), d^2(x_{2n}, x_{2n+1}), p^2(z, Az)\} \\
&\quad + c_2 \max\{d(x_{2n}, x_{2n+1})p(x_{2n}, Az), d(z, x_{2n+1})p(z, Az)\} \\
&\quad + c_3 p(x_{2n}, Az)d(z, x_{2n+1})
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lim_{n\to\infty} D^2(Ax_{2n}, Az) &\leq \lim_{n\to\infty} [c_1 \max\{d^2(x_{2n}, z), d^2(x_{2n}, x_{2n+1}), p^2(z, Az)\} \\
&\quad + c_2 \max\{d(x_{2n}, x_{2n+1})p(x_{2n}, Az), d(z, x_{2n+1})p(z, Az)\} \\
&\quad + c_3 p(x_{2n}, Az)d(z, x_{2n+1})]
\end{aligned}$$

$$\begin{aligned}
D^2(z, Az) &\leq c_1 \max\{d^2(z, z), d^2(z, z), p^2(z, Az)\} \\
&\quad + c_2 \max\{d(z, z)p(z, Az), d(z, z)p(z, Az)\} \\
&\quad + c_3 p(z, Az)d(z, z)
\end{aligned}$$

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$\Longrightarrow D^2(z, Az) \leq c_1 p^2(z, Az)$

$\therefore D(z, Az) \leq (c_1)^{1/2} p(z, Az).$

Now, equation (3.6), implies that,

$p\alpha(z, Az) \leq (c_1)^{1/2} p(z, Az).$

Since $(c_1)^{1/2} < 1$, so we get $p(z, Az) = 0$. Similarly we prove that $p(z, Bz) = 0$. That is, there exist $z \in X$, such that $\{z\} \subset Az$ and $\{z\} \subset Bz$. Also we know that $BAz \subset Bz$ and $ABz \subset Az$ and from equation (3.5) there exists $z \in X$ such that $\{z\} \subset ABz \subset Az$ and $\{z\} \subset BAz \subset Bz$.

ii) Priori error estimation:

From triangle inequality of metric and equation 3.4,

$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p})$

$\leq (c_1+2c_2)^{1/2} d(x_{n-1}, x_n) + (c_1+2c_2)^{1/2} d(x_n, x_{n+1})$

$\ldots + (c_1+2c_2)^{1/2} d(x_{n+p-2}, x_{n+p-1})$

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$\leq (c_1 + 2c_2)^{n/2} d(x_0, x_1) + (c_1 + 2c_2)^{(n+1)/2} d(x_0, x_1)$

$+ \ldots + (c_1 + 2c_2)^{(n+p-1)/2} d(x_0, x_1)$

$$d(x_n, x_{n+p}) \leq \sum_{k=0}^{p-1} (c_1 + 2c_2)^{(n+k)/2} d(x_0, x_1).$$

$$\leq \lim_{p\to\infty} \sum_{k=0}^{p-1} (c1 + 2c2)^{(n+k)/2} d(x0, x1)$$

$$\leq \sum_{k=0}^{\infty} (c1 + 2c2)^{(n+k)/2} d(x0, x1)$$

iii) Posteriori error estimation:
From triangle inequality of metric and equation 3.4,

$$d(xn, xn+q) \leq d(xn, xn+1) + d(xn+1, xn+2) + . . . + d(xn+q-1, xn+q)$$

$$\leq d(xn, xn+1) + (c1+2c2)^{1/2} d(xn, xn+1)$$ ________

$$. . . + (c1+2c2)^{1/2} d(xn+q-2, xn+q-1)$$

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$$\leq d(xn, xn+1) + (c1 + 2c2)^{(n+1)/2} d(xn, xn+1)$$

$$+ . . . + (c1 + 2c2)^{(p-1)/2} d(xn, xn+1)$$

$$d(xn, xn+q) \leq n \sum_{k=0}^{q-1} (c1 + 2c2)^{k/2} d(xn, xn+1).$$

$$\Rightarrow \lim_{q\to\infty} d(xn, xn+q) \leq \lim_{q\to\infty} \sum_{k=0}^{q-1} (c1 + 2c2)^{k/2} d(xn, xn+1)$$

$$d(xn, z) \leq \sum_{k=0}^{\infty} (c1 + 2c2)^{k/2} d(xn, xn+1)$$

iv) Now, we establish the rate of convergence of fuzzy mappings A, B as follows; for an even number $n \in N$,

$d(xn, z) = d(xn, BAz)$

$d(BAxn-2, BAz)$

$(c1 + 2c2)d(xn-2, BAz)$

$(c1 + 2c2)d(BAxn-4, BAz)$

$(c1 + 2c2)^2 d(BAxn-6, BAz)$

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$\leq (c1 + 2c2)^{n/2} d(BAx0, BAz)$,

and for an odd number $n \in N$,

$d(x_n, z) = d(x_n, ABz)$

$d(ABx_{n-2}, ABz)$

$(c_1 + 2c_2)d(x_{n-2}, ABz)$

$(c_1 + 2c_2)d(ABx_{n-4}, ABz)$

$(c_1 + 2c_2)^2 d(ABx_{n-6}, ABz)$

.

.

$\leq (c_1 + 2c_2)^{(n-1)/2}\, d(ABx_1, ABz)$,

$$\Rightarrow \quad d(x_n, z) \leq \begin{cases} (c_1 + 2c_2)^{n/2}\, d(BAx_0, BAz), & \text{if n is even number.} \\ (c_1 + 2c_2)^{(n-1)/2}\, d(ABx_1, ABz), & \text{if n is odd number.} \end{cases}$$

Since, for any $N \in N$, $\{x_{2N+1}\} \subset ABx_{2N-1}$ and $\{x_{2N}\} \subset ABx_{2N-2}$,

$$\therefore \quad d(x_n, z) \leq (c_1 + 2c_2)^{(n-1)/2}\, d(ABx_1, ABz)$$

$$\leq (c_1 + 2c_2)^{n/2}\, d(BAx_0, BAz)$$

$\leq (c_1 + 2c_2)^{n/2}\, d(x_0, z)$.
for $n = 0, 1, 2, 3, \ldots$, which proves the rate of convergence. □

Example 1: Let $X = [0, 1]$ be a metric space with the metric $d(x, y) = |x - y|$, $\forall\ x, y \in X$. Define fuzzy mappings A, B from X into W(X), such that for any $x \in X$, $Ax$ is a characteristic functions for $\{(3/4)\, x\}$ and $Bx$ is a characteristic functions for $\{x^2\}$. Assume $x_0 = 1 \in X$, then,

$$\{3/4\} = \{x_1\} \subset Ax_0$$
$$\{(3/4)^2\} = \{x_2\} \subset Bx_1,$$
$$\vdots$$
$$\{(3/4)^{2(2n-1)}\} = \{x_{2n}\} \subset \{B_{2n-1}\},$$
$$\{(3/4)^{2(2n-(1/2))}\} = \{x_{2n+1}\} \subset \{A_{2n}\}, \ \ldots .$$

for $n = 0, 1, 2, 3, \ldots$.

This implies that, for each $x, y \in X$, we can find $c_1 = (9/16)$, $c_2 = 0$, $c_3<1$,

(or $c_1 = 0$, $c_2 = \frac{9}{32}$, $c_3 < \frac{3^2}{32}$) with $c_1 + 2c_2 < 1$ & $c_2 + c_3 < 1$, such that

$$D^2(Ax, By) \leq c_1 \max\{d^2(x, y), p^2(x, Ax), p^2(y, By)\}$$

$$+c_2 \max\{p(x, Ax)p(x, By), p(y, Ax)p(y, By)\}$$
$$+ c_3 p(x, By)p(y, Ax).$$

Hence the characteristic function for {0} is the common fixed point of A and B in W (X).
Priori error:

$$d(x_n, z) \leq \frac{(c_1)^{n/2}}{1-(c_1)^{1/2}} \varepsilon_0 = (9/16)^{n/2}, \quad \text{for } n = 1, 2, 3, \ldots$$

where $\varepsilon_0 = d(x_0, x_1)$.

Posteriori error:

$$d(x_n, z) \leq \frac{\varepsilon_n}{1-(c_1)^{1/2}} = 4\varepsilon_n, \quad \text{for } n = 1, 2, 3, \ldots$$

where $\varepsilon_n = d(x_n, x_{n+1})$.

Rate of convergence:

$$d(x_n, z) \leq (c_1)^{n/2} d(x_0, z) = (9/16)^{n/2}.$$

Remark 1: In the above example, if $c_1 = 0$, $c_2 = \frac{9}{32}$ and $c_3 < \frac{9}{32}$ then also,

Priori error, Posteriori error and Rate of convergence are remains the same.
Remark 2: In the process of simplifying the contraction equation 3.7, either $p(x, By) = 0$ or $p(y, Ax) = 0$. So, by theorem 2, we can establish the following corollary.
Corollary 1: Let (X, d) be a complete metric space, A, B : X → W (X) be fuzzy mappings. Assume that there exist $c_1, c_2 \in I$ with $c_1 + c_2 < 1$, such that for all $x, y \in X$

$$D^2(Ax, By) \leq c_1 \max\{d^2(x, y), p^2(x, Ax), p^2(y, By)\}$$

$$c_2 \max\{p(x, Ax)p(x, By), p(y, Ax)p(y, By)\}. \quad (3.7)$$

Then, there exists $z \in X$, such that $\{z\} \subset Az$ and $\{z\} \subset Bz$.

The proof of above corollary follows the proof of theorem 2, also, the error estimations and rate of convergence are same.

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