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# Homotopy Methods for Solving Kaup-Boussinesq System

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**Abstract- HAM and HPM are implemented for solving Kaup-Boussinesq system. The two methods are successfully applied for solving this kind of problems. Both were very convenient and accurate. The approximate solution is considered as an infinite series usually converge to the exact solution by these methods. As a result, we have seen that the efficiency and accuracy of HAM is more than HPM because it contains auxiliary parameter $\hbar$, that is why HAM considered to be more general.**
**Keywords – Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM), Kaup-BoussinesqSystem (KB).**

## I. INTRODUCTION:

Partial differential equations (PDEs) are widespread in most fields of science including: engineering, Finance and economics. The natural phenomena and model multidimensional dynamical systems are often described by PDEs [1].

Most of these problems do not have analytical solutions excluding a partial number of them. While the most challenging task for this class of equation is finding analytical solutions. Recently, there are various suggestions of methods for the analytical solutions of nonlinear differential equations which might not be influenced by the existence of small or large parameter in the equation [13]. A graphic example of this is Homotopy analysis method (HAM) [22]Another example as [9] mentioned is homotopy perturbation method.

[15]states that Homotopy method is one of the most resent common techniques, which is a collection of the classical perturbation technique and homotopy concept that used in topology. Moreover, this technique has been firstly proposed by [22], named the homotopy analysis method (HAM). Then later this method was used by several authors for solving differential equations. One of the benefits of analytical method over perturbation methods is that this method is not relying on small or large parameters. Furthermore, all nonlinear equations cannot be solved by perturbation methods, because they are based on existence of small or large parameter[14]. However, unperturbation methods including, $\delta -$expansion and ADM, have independency over small parameters. Secondly, it can guarantee that the convergence of solution series in the convenient way by applying HAM, and by this aspect it differs from other analytical techniques. Moreover, as it based on homotopy in topology, it is giving as freedom in selecting equation types of linear sub-problems, base function of solution, initial guess and so on, that is why many sophisticated types of equations such as: ODEs and PDEs often solves in a simple way [21].

J.H.He[9] has proposed and developed Homotopy perturbation method. As it is a combination of the traditional perturbation method and homotopy in topology, it can be changed continuously to a soft problem, which can be solved in an easy way. And because it is known as an effective and reliable method, it can be used in solving differential, ordinary, partial and integral equation, linear or nonlinear such as: [18, 4, 17, 3]. And also as a powerful method, it considers the approximate solution of nonlinear problems as an infinite series usually converge to the exact solution.

The Taylor series is a base of the both methods with respecting to an embedding parameter. Moreover, if auxiliary linear operator and initial guess are good enough, by a few terms both methods can give extremely good approximation. And because the HAM is fundamentally contain the auxiliary parameter $\hbar$, which lets us to adjust and control the convergence region and rate of solution series in an easy way, that is why it is not necessarily for the HAM to contain a good initial guess, while this aspect differs it from HPM, because HPM had to use a good enough initial guess. So, the homotopy analysis method is more general [20].

A completely integrable system of nonlinear partial differential equations is called Kaup-Bousinesq system [23]. Is a model for long waves propagating at the surface of perfect fluid [2].also KB is one of the hydrodynamical models as it arise in the theory of water waves [23].

$$\left.\begin{aligned} u_t - v_{xxx} - 2(vu)_x &= 0, \\ v_t - u_x - 2vv_x &= 0. \end{aligned}\right\} \tag{1}$$

With the initial conditions:
$u(x,0)=\frac{w^2}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right)-\frac{w^2}{4}\left(1+\tanh\left(\frac{wx}{2}\right)\right)^2$,
And
$v(x,0)=\frac{-w}{2}\left(1+\tanh\left(\frac{wx}{2}\right)\right)$.
Where $u=u(x,t)$ indicate to the height of the water surface above a horizontal bottom, $v=v(x,t)$ is related to the horizontal velocity field and $w$ is constant.
It is called the Kaup-Boussinesq system because they have been used Boussinesq scaling in the derivation, and it has been studying by Kaup[6]. It has also been used by L. J. F. Broer[12]. Also as it goes to the family of long-waves models established by Boussinesq, drawn-out by [5, 16] and many others.
In recent years, the KB system has been the subject for many other researches. [8] Work on Solitary-wave solution to a dual equation of the KB system. And [7] work on travelling wave solution of nonlinear systems of PDEs by using the factional variable method.
It has been proven that the HAM and HPM are powerful and convenient methods in their applications. The aim of this paper is using HAM and HPM numerically for solving Kaup-Boussinesq system, and compared the both methods with the exact solution. Also the accuracy of the presented methods at different values of $x$ and fixed time $t$ was discussed.

## II. PROPOSED ALGORITHM

*2.1 Basic idea of Homotopy perturbation method-*
To clarify the basic concept of thehomotopy perturbation method, consider the following nonlinear differential equation [10]

$$A(u)-f(r)=0,\ r\in\delta \tag{2}$$

With boundary conditions:

$$B(u,\partial u/\partial n)=0,\ r\in\tau \tag{3}$$

Where $A$ is a general differential operator, $f(r)$is a known analytical function, $B$ is a boundary operator and $\tau$ is the boundary of the domain $\delta$.The operator $A$ can be divided into two parts $L$and $N$, where $L$ is a linearand $N$ is a nonlinear operator. Therefore (2) can be written as follow:

$$L(u)+N(u)-f(r)=0.$$

We construct a homotopy$v(r,p):\delta\times[0,1]\rightarrow\mathbb{R}$, which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(r)]=0,\ p\in[0,1],\ r\in\delta, \tag{4}$$

In which $p\in[0,1]$ is called homotopy parameter and$u_0$ is an initial approximation of (2) satisfying the given conditions. From (2), we have

$$H(v,0)=L(v)-L(u_0)=0,$$
$$H(v,1)=A(v)-f(r)=0.$$

The changing process of $p$ from zero to unity is just that of $v(r,p)$ from $u_0(r)$to $u(r)$. In topology, this is called deformation.And $L(v)-L(u_0)$, $A(v)-f(r)$ are called homotopic. According to HPM, as a small parameterwe can firstly use the embedding parameter $p$, and assume that the Solution of (4) can be written as a power series in $p$ as follows:

$$v=v_0+pv_1+p^2v_2+\cdots.$$

Putting $p=1$results in the approximate solution in the form of:

$$V=\lim_{p\to1}v=v_0+v_1+v_2+\cdots.$$

[11] has proved the convergence of the above equation.

*2.2 Basic idea of Homotopy analysis method-*
To illustrate the main idea of the homotopy analysis method we consider the following nonlinear equation.

$\mathcal{N}\big(u(x,t)\big) = 0,$

Where $\mathcal{N}$ is a nonlinear operator, $x$ and t indicate as independent variables, $u(x,t)$ is unknown function. The so-called a zero-order deformation was constructed by [19]:

$$(1-q)\ell\big(\varphi(x,t;q) - u_0(x,t)\big) = q\hbar\mathcal{H}(x,t)\mathcal{N}\big(\varphi(x,t;q)\big), \tag{5}$$

Where $\hbar \neq 0$ denote an auxiliary parameter, $q \in [0,1]$ is an embedding parameter, $\mathcal{H}(x,t)$ is an auxiliary function, $\ell$ is an auxiliary linear operator, $\varphi(x,t;q)$ is unknown function and $u_0(x)$ is an initial guess of $u(x,t)$. The importance of the HAM is in that one has freedom to choose auxiliary parameter $\hbar$. If $q = 0$ and $q = 1$, it holds

$$\varphi(x,t;0) = u_0(x), \qquad \varphi(x,t;1) = u(x,t) \tag{6}$$

Thus as $q$ increases from $0$to $1$, the solute`on$\varphi(x,t;q)$ varies from the initial guess $u_0(x)$ to the solution $u(x,t)$. Expanding $\varphi(x,t;q)$ in Taylor series with respect to $q$, one has

$$\varphi(x,t;q) = u_0(x) + \sum_{K=1}^{\infty} u_K(x,t)q^K, \tag{7}$$

Where

$$u_k(x,t) = \left.\frac{\partial^k \varphi(x,t;q)}{\partial q^k}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary $\mathcal{H}(x,t)$ are so properly chosen, then the series (7) converges at $q = 1$, one has

$$\varphi(x,t;1) = u_0(x) + \sum_{k=1}^{\infty} u_k(x,t) \tag{8}$$

When we take $\hbar = -1$ and $\mathcal{H}(x,t) = 1$ the equation (5) becomes:

$$(1-q)\ell\big(\varphi(x,t;q) - u_0(x)\big) + q\mathcal{N}\big(\varphi(x,t;q)\big) = 0,$$

This is mostly used in HPM.

According to (8), the governing equation and the corresponding initial condition of $u_k(x,t)$ can be deduced from the zero-order deformation, (5) define the vector

$$\vec{u}_n(x,t) = \{u_0(x), u_1(x,t), \ldots, u_n(x,t)\}.$$

Differentiating (5) $k$ times with respect to the embedding parameter $q$ and then setting $q = 0$ and finally dividing them by $k!$, we have the so-called $k$th-order deformation equation

$$\ell[u_k(x,t) - \mathcal{X}_k u_{k-1}(x,t)] = \hbar\mathcal{H}(x,t)\mathcal{R}_k[\vec{u}_{k-1}(x,t)] \tag{9}$$

Where

$$\mathcal{R}_k[\vec{u}_{k-1}(x,t)] = \frac{1}{(k-1)!}\left.\frac{\partial^{k-1}\mathcal{N}\big(\varphi(x,t;q)\big)}{\partial q^{k-1}}\right|_{q=0},$$

And

$$\mathcal{X}_k = \begin{cases} 0, & k \leq 1 \\ 1, & k > 1 \end{cases} \tag{10}$$

It should be emphasized that $u_k(x,t)$ for $k \geq 1$ is governed the linear equation (9) with the linear boundary conditions that comes from the original problem, which can be easily solved, especially by means of symbolic computation software such as Matlab.

## III. EXPERIMENT AND RESULT

The following example is solved numerically by the presented methods:

$$u_t - v_{xxx} - 2(vu)_x = 0,$$

$$v_t - u_x - 2vv_x = 0.$$

With the initial points:

$$u(x,0) = \frac{w^2}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1 + \tanh\left(\frac{wx}{2}\right)\right)^2,$$

And

$$v(x,0) = \frac{-w}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right).$$

Where $w = 1.5$, and with the soliton solutions:
$u(x,t) = \frac{w^2}{2}\left(1 + \tanh\left(\frac{w(x - wt)}{2}\right)\right) - \frac{w^2}{4}\left(1 + \tanh\left(\frac{w(x - wt)}{2}\right)\right)^2$
And
$v(x,t) = \frac{-w}{2}\left(1 + \tanh\left(\frac{w(x-wt)}{2}\right)\right)$.

*3.1 The solution of the Kaup-Boussinesq system by HPM-*

For solving system (1) by HPM we construct the following homotopies:
$H_1(U,V,p) = (1 - p)[L(U) - L(u_0)] + p[A(U,V) - f(r)] = 0,$

And
$H_2(U,V,p) = (1 - p)[L(V) - L(v_0)] + p[A(U,V) - f(r)] = 0.$

$L(U) = \frac{\partial U}{\partial t}$, $L(V) = \frac{\partial V}{\partial t}$, $L(u_0) = \frac{\partial u_0}{\partial t}$,
$u_0 = u(x,0)$, $v_0 = v(x,0)$.
Then

$$H_1(U,V,p) = (1 - p)\left(\frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left[\frac{\partial U}{\partial t} - \frac{\partial^3 V}{\partial x^3} - 2V\frac{\partial U}{\partial x} - 2U\frac{\partial V}{\partial x}\right] = 0, \qquad (11)$$

And

$$H_2(U,V,p) = (1 - p)\left(\frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t}\right) + p\left[\frac{\partial V}{\partial t} - \frac{\partial U}{\partial x} - 2V\frac{\partial V}{\partial x}\right] = 0. \qquad (12)$$

Where
$U(x,t_0) = u(x,t_0)$,
And
$V(x,t_0) = v(x,t_0)$.
Let's present the solution of the system (11), (12) as the following:
$U = U_0 + pU_1 + p^2U_2 + \cdots.$
And
$V = V_0 + pV_1 + P^2V_2 + \cdots.$
And by substituting the above equations into (11) and (12), respectively, we get:

$$H_1(U,V,p) = \frac{\partial U}{\partial t} - \frac{\partial u_0}{\partial t} + p\frac{\partial u_0}{\partial t} - p\frac{\partial^3 V}{\partial x^3} - 2pV\frac{\partial U}{\partial x} - 2pU\frac{\partial V}{\partial x} = 0,$$

$$H_1(U,V,p) = \frac{\partial}{\partial t}(U_0 + pU_1 + p^2U_2 + \cdots) - \frac{\partial u_0}{\partial t} + p\frac{\partial u_0}{\partial t} - p\frac{\partial^3}{\partial x^3}(V_0 + pV_1 + p^2V_2 + \cdots)$$
$$-2p(V_0 + pV_1 + p^2V_2 + \cdots)\frac{\partial}{\partial x}(U_0 + pU_1 + p^2U_2 + \cdots)$$
$$-2p(U_0 + pU_1 + p^2U_2 + \cdots)\frac{\partial}{\partial x}(V_0 + pV_1 + p^2V_2 + \cdots) = 0,$$

And

$$H_2(U,V,p) = \frac{\partial V}{\partial t} - \frac{\partial v_0}{\partial t} + p\frac{\partial v_0}{\partial t} - p\frac{\partial U}{\partial x} - 2pV\frac{\partial V}{\partial x} = 0,$$

$$H_2(U,V,p) = \frac{\partial}{\partial t}(V_0 + pV_1 + p^2V_2 + \cdots) - \frac{\partial v_0}{\partial t} + p\frac{\partial v_0}{\partial t} - p\frac{\partial}{\partial x}(U_0 + pU_1 + p^2U_2 + \cdots)$$
$$-2p(V_0 + pV_1 + p^2V_2 + \cdots)\frac{\partial}{\partial x}(V_0 + pV_1 + p^2V_2 + \cdots) = 0.$$

By equating the same powers of $p$, we get:

$$p^0: \frac{\partial U_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \qquad (13)$$

$$p^0: \frac{\partial V_0}{\partial t} - \frac{\partial v_0}{\partial t} = 0 \tag{14}$$

$$p^1: \frac{\partial U_1}{\partial t} + \frac{\partial u_0}{\partial t} - \frac{\partial^3 V_0}{\partial x^3} - 2V_0 \frac{\partial U_0}{\partial x} - 2U_0 \frac{\partial V_0}{\partial x} = 0 \tag{15}$$

$$p^1: \frac{\partial V_1}{\partial t} + \frac{\partial v_0}{\partial t} - \frac{\partial U_0}{\partial x} - 2V_0 \frac{\partial V_0}{\partial x} = 0 \tag{16}$$

$$p^2: \frac{\partial U_2}{\partial t} - \frac{\partial^3 V_1}{\partial x^3} - 2V_1 \frac{\partial}{\partial x} U_0 - 2V_0 \frac{\partial}{\partial x} U_1 - 2U_1 \frac{\partial}{\partial x} V_0 - 2U_0 \frac{\partial}{\partial x} V_1 = 0 \tag{17}$$

$$p^2: \frac{\partial V_2}{\partial t} - \frac{\partial}{\partial x} U_1 - 2V_0 \frac{\partial}{\partial x} V_1 - 2V_1 \frac{\partial}{\partial x} V_0 = 0 \tag{18}$$

By the same way find $p^3$ and so on.
By integrating both side of the equation (13), we get:

$U_0 = u_0 = u(x, 0),$

Same for equation (14), we get:

$V_0 = v_0 = v(x, 0),$

And from (15) by integrating both sides of the equation we get:
$U_1(x,t) = \int_0^t \left( \frac{\partial^3 V_0(x)}{\partial x^3} + 2V_0(x) \frac{\partial U_0(x)}{\partial x} + 2U_0(x) \frac{\partial V_0(x)}{\partial x} \right) ds$,

Then
By integrating both sides of the equation (16) we get:
$V_1(x,t) = \int_0^t \left( \frac{\partial U_0(x)}{\partial x} + 2V_0(x) \frac{\partial V_0(x)}{\partial x} \right) ds$,

After we have found $U_1(x,t)$ and $V_1(x,t)$, we have to put them into (17) and (18) to find $U_2(x,t)$, $V_2(x,t)$ and so on.
Thus, we can obtain
$u(x,t) = U_0 + U_1 + U_2 + \cdots.$
Also
$v(x,t) = V_0 + V_1 + V_2 + \cdots.$

*3.2 Solution of the Kaup-Boussinesq system by HAM-*

To solve the KB system by HAM, which was constructed by Liao [22]. First we will start with the following deformation equations,

$$(1-q)\ell_u[\varphi(x,t;q) - u_0(x,t)] = q\hbar\mathcal{H}_1(x,t)\mathcal{N}_1[\varphi(x,t;q)], \tag{19}$$

$$(1-q)\ell_v[\psi(x,t;q) - v_0(x,t)] = q\hbar\mathcal{H}_2(x,t)\mathcal{N}_2[\psi(x,t;q)], \tag{20}$$

Where $\mathcal{N}_1$, $\mathcal{N}_2$ are two nonlinear operators, $x$ and $t$ indicate to be independent variables, $\mathcal{H}_1(x,t)$ and $\mathcal{H}_2(x,t)$ are auxiliary functions, $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $u_0(x)$ and $v_0(x)$ are initial guesses of $u(x,t)$ and $v(x,t)$, respectively. The functions $\varphi(x,t;q)$ and $\psi(x,t;q)$ are known functions and $\ell_u$ and $\ell_v$ are auxiliary operators that are defined as follows:
$\ell_u[u(x,t)] = \frac{\partial u}{\partial t}$, $\ell_v[v(x,t)] = \frac{\partial v}{\partial t}$.
Which satisfies:
$\ell_u[C_1(x)] = 0$, $\ell_v[C_2(x)] = 0$.
Where $C_1(x)$ and $C_2(x)$ are integral constants. When $q = 0$ and $q = 1$, we get:
$\varphi(x,t;0) = u_0(x)$, $\varphi(x,t;1) = u(x,t)$,
$\psi(x,t;0) = v_0(x)$, $\psi(x,t;1) = v(x,t)$.

As $q$ increase from $0$ to $1$, the solution of KB will be vary from the initial guesses $u_0(x)$ and $v_0(x)$ to the exact solution $u(x,t)$ and $v(x,t)$ expanding $\varphi(x,t;q)$ and $\psi(x,t;q)$ as a Taylor series with respect to $q$, gives
$\varphi(x,t;q) = \varphi(x,t;0) + \sum_{k=1}^{\infty} \frac{q^k}{k!} \frac{\partial^k \varphi(x,t;q)}{\partial q^k}\Big|_{q=0}$,

$\psi(x,t;q) = \psi(x,t;0) + \sum_{k=1}^{\infty} \frac{q^k}{k!} \frac{\partial^k \psi(x,t;q)}{\partial q^k}\Big|_{q=0}$,

Thus

$\varphi(x,t;q) = u_0(x) + \sum_{k=1}^{\infty} u_k(x,t) q^k$ (21)

$\psi(x,t;q) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,t) q^k$ (22)

Where

$u_k(x,t) = \frac{1}{k!} \frac{\partial^k \varphi(x,t;q)}{\partial q^k}\Big|_{q=0}$ (23)

$v_k(x,t) = \frac{1}{k!} \frac{\partial^k \psi(x,t;q)}{\partial q^k}\Big|_{q=0}$ (24)

If $q = 1$

$\varphi(x,t;1) = u(x,t) = u_0(x) + \sum_{k=1}^{\infty} u_k(x,t)$ (25)

$\psi(x,t;1) = v(x,t) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,t)$ (26)

Which should be one of the solutions of coupled KB system (1). For further analysis, the vectors

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \dots, u_n(x,t)\},$$

$$\vec{v}_n(x,t) = \{v_0(x,t), v_1(x,t), \dots, v_n(x,t)\},$$

Are defined. Differentiating equations (19) and (20) $k$-times with respect to the parameter $q$ and then setting $q = 0$ and finally dividing them by $k!$ we have the kth-order deformation equations:.

$\ell_u[u_k(x,t) - \mathcal{X}_k u_{k-1}(x,t)] = \hbar \mathcal{H}_1(x,t) \mathcal{R}1_k[\vec{u}_{k-1}]$ (27)

$\ell_v[v_k(x,t) - \mathcal{X}_k v_{k-1}(x,t)] = \hbar \mathcal{H}_2(x,t) \mathcal{R}2_k[\vec{v}_{k-1}]$ (28)

With the initial conditions

$u_k(x,0) = 0$,

$v_k(x,0) = 0$,

Now, while applying this method on the KB system when the nonlinear operators as we defined before are:

$\mathcal{N}_1(\varphi) = \varphi_t - \psi_{xxx} - 2\psi\varphi_x - 2\varphi\psi_x$,

$\mathcal{N}_2(\psi) = \psi_t - \phi_x - 2\psi\psi_x$,

Where $\varphi = \varphi(x,t;q)$, $\psi = \psi(x,t;q)$, by using the above definition, we construct the equations (19), (20), (27) and (28).

Now the solution of the $k$th order deformation equation are

$u_k(x,t) = \mathcal{X}_k u_{k-1}(x,t) + \hbar L_t^{-1}[\mathcal{R}1_k(\vec{u}_{k-1})]$ (29)

$v_k(x,t) = \mathcal{X}_k v_{k-1}(x,t) + \hbar L_t^{-1}[\mathcal{R}2_k(\vec{v}_{k-1})]$ (30)

$k \geq 1$, and selecting $\mathcal{H}_1(x,t) = 1$, $\mathcal{H}_2(x,t) = 1$

Where $\mathcal{X}_k$ is defined in the equation (10)

While

$$[\mathcal{R}1_k(\vec{u}_{k-1})] = \frac{\partial u_{k-1}(x,t)}{\partial t} - \frac{\partial^3 v_{k-1}(x,t)}{\partial x^3} - 2\sum_{i=0}^{k-1} v_i(x,t) \frac{\partial}{\partial x} u_{k-1-i}(x,t)$$
$$-2\sum_{i=0}^{k-1} u_i(x,t) \frac{\partial}{\partial x} v_{k-1-i}(x,t) \quad (31)$$

And

$[\mathcal{R}2_k(\vec{v}_{k-1})] = \frac{\partial v_{k-1}(x,t)}{\partial t} - \frac{\partial u_{k-1}(x,t)}{\partial x} - 2\sum_{i=0}^{k-1} v_i(x,t) \frac{\partial}{\partial x} v_{k-1-i}(x,t)$ (32)

Put $k = 1$ in equations (29) and (30) and start with the initial approximations $u_0(x)$ and $v_0(x)$ to get $u_1(x,t)$ and $v_1(x,t)$ as:

$u_1(x,t) = \hbar L_t^{-1}[\mathcal{R}1_1(\vec{u}_0)]$ (33)

$$v_1(x,t) = \hbar L_t^{-1}[\mathcal{R}2_1(\vec{v}_0)] \tag{34}$$

Put $k = 2$ in to the equation (29) and (30) by using $u_1(x,t)$ and $v_1(x,t)$ we will get:

$$u_2(x,t) = u_1(x,t) + \hbar L_t^{-1}[\mathcal{R}1_2(\vec{u}_1)] \tag{35}$$

$$v_2(x,t) = v_1(x,t) + \hbar L_t^{-1}[\mathcal{R}2_2(\vec{v}_1)] \tag{36}$$

By the same way find $u_3(x,t), v_3(x,t)$ and so on

Then the approximate solution of equation (1) by HAM is:

$$u(x,t) = u_0(x) + \sum_{k=1}^{\infty} u_k(x,t),$$
$$= u_0(x) + u_1(x,t) + \cdots.$$

And

$$v(x,t) = v_0(x) + \sum_{k=1}^{\infty} v_k(x,t),$$
$$= v_0(x) + v_1(x,t) + \cdots.$$

*3.3 Applying the Homotopy perturbation method-*

By applying HPM to the example we get:

$$p^0: \frac{\partial U_0}{\partial t} = \frac{\partial u_0}{\partial t}, U_0(x) = u_0(x),$$

Then

$$U_0(x) = \frac{w^2}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right) - \frac{w^2}{4}\left(1 + \tanh\left(\frac{wx}{2}\right)\right)^2.$$

For finding $V_0(x)$ when we have:

$$p^0: \frac{\partial V_0}{\partial t} = \frac{\partial v_0}{\partial t}, V_0(x) = v_0(x),$$

Then

$$V_0(x) = \frac{-w}{2}\left(1 + \tanh\left(\frac{wx}{2}\right)\right).$$

For finding $U_1(x,t)$ and $V_1(x,t)$ take:

$$p^1: \frac{\partial U_1}{\partial t} = -\frac{\partial u_0}{\partial t} + \frac{\partial^3 V_0}{\partial x^3} + 2V_0\frac{\partial U_0}{\partial x} + 2U_0\frac{\partial V_0}{\partial x},$$

By taking integral of both sides:

$$\frac{\partial u_0}{\partial t} = 0,$$

$$U_1(x,t) = \int_0^t \left(\frac{\partial^3 V_0(x)}{\partial x^3} + 2V_0(x)\frac{\partial U_0(x)}{\partial x} + 2U_0(x)\frac{\partial V_0(x)}{\partial x}\right) ds,$$

Then

$$U_1(x,t) = \frac{W^4}{8}\operatorname{sech}^4\left(\frac{wx}{2}\right)t + \frac{W^4}{4}\tanh\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t - \frac{W^4}{8}\operatorname{sech}^2\left(\frac{wx}{2}\right)t + \frac{W^4}{8}\tanh^2\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t$$

And

$$p^1: \frac{\partial V_1}{\partial t} = -\frac{\partial v_0}{\partial t} + \frac{\partial U_0}{\partial x} + 2V_0\frac{\partial V_0}{\partial x},$$

By integrating both sides we get:

$$\frac{\partial v_0}{\partial t} = 0,$$

$$V_1(x,t) = \frac{w^3}{4}\operatorname{sech}^2\left(\frac{wx}{2}\right)t.$$

By the same way find $U_2, V_2, \cdots$.

Then the approximate solutions of the HPM are:

$$u(x,t) = U_0(x) + U_1(x,t) + U_2(x,t) + \cdots.$$
$$v(x,t) = V_0(x) + V_1(x,t) + V_2(x,t) + \cdots.$$

*3.4 Applying the Homotopy analysis method-*

When $k = 1$ put in (31) and (32) to find $u_1(x,t)$ and $v_1(x,t)$:

$$[\mathcal{R}1_1(u_0)] = \frac{\partial u_0(x)}{\partial t} - \frac{\partial^3 v_0(x)}{\partial x^3} - 2v_0(x)\frac{\partial u_0(x)}{\partial x} - 2u_0(x)\frac{\partial v_0(x)}{\partial x},$$

And

$$[\mathcal{R}2_1(v_0)] = \frac{\partial v_0(x)}{\partial t} - \frac{\partial u_0(x)}{\partial x} - 2v_0(x)\frac{\partial v_0(x)}{\partial x}.$$

While $\frac{\partial u_0(x)}{\partial t} = 0$, and $\frac{\partial v_0(x)}{\partial t} = 0$,

Then

$$[\mathcal{R}1_1(u_0)] = -\left[\frac{w^4}{8}\operatorname{sech}^4\left(\frac{wx}{2}\right) + \frac{w^4}{4}\tanh\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right) - \frac{w^4}{8}\operatorname{sech}^2\left(\frac{wx}{2}\right) + \frac{w^4}{8}\tanh^2\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)\right],$$

And from (33) we get:

$$u_1(x,t) = (-\hbar)\int_0^t\left(\frac{\partial^3 v_0(x)}{\partial x^3} + 2v_0(x)\frac{\partial u_0(x)}{\partial x} + 2u_0(x)\frac{\partial v_0(x)}{\partial x}\right)ds,$$

Then

$$u_1(x,t) = (-\hbar)\left[\frac{w^4}{8}\operatorname{sech}^4\left(\frac{wx}{2}\right)t + \frac{w^4}{4}\tanh\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t - \frac{w^4}{8}\operatorname{sech}^2\left(\frac{wx}{2}\right)t + \frac{w^4}{8}\tanh^2\left(\frac{wx}{2}\right)\operatorname{sech}^2\left(\frac{wx}{2}\right)t\right]$$

And

$$[\mathcal{R}2_1(v_0)] = -\left[\frac{w^3}{4}\operatorname{sech}^2\left(\frac{wx}{2}\right)\right].$$

Then from (34) we get:

$$v_1(x,t) = (-h)\int_0^t\left(\frac{\partial u_0(x)}{\partial x} + 2v_0(x)\frac{\partial v_0(x)}{\partial x}\right)ds,$$

$$v_1(x,t) = (-\hbar)\left[\frac{w^3}{4}\operatorname{sech}^2\left(\frac{wx}{2}\right)t\right].$$

By the same way find $u_2(x,t)$, $v_2(x,t)$ and so on.

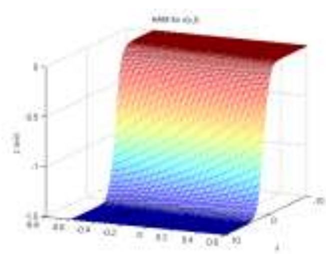

Figure (1): Solution for $v(x,t)$ by HAM

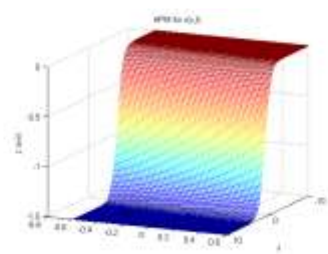

Figure (2): Solution for $v(x,t)$ by HPM

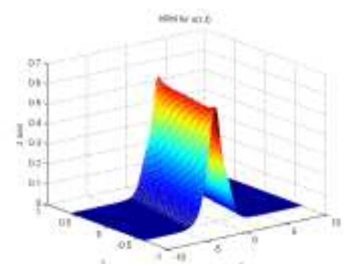

Figure (3): Solution for $u(x,t)$ by HPM

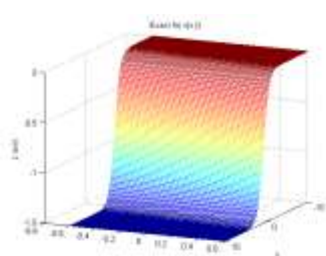

Figure (4):Exact solution for $v(x,t)$

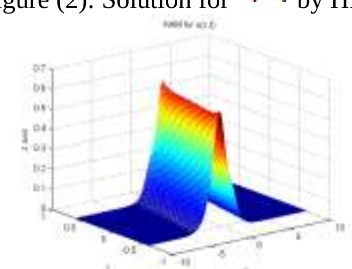

Figure (5): Solution for $u(x,t)$ by HAM

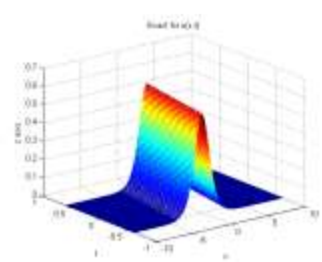

Figure (6):Exact solution for $u(x,t)$

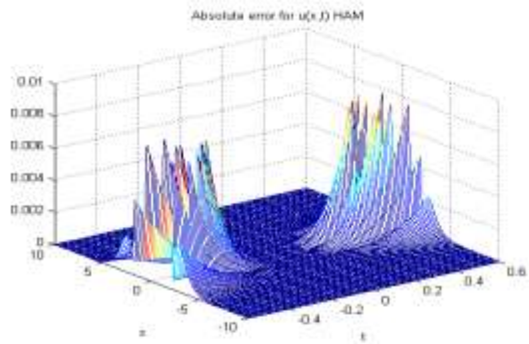


Figure (7): Absolute errors between Exact solution and HAM for $u(x,t)$.

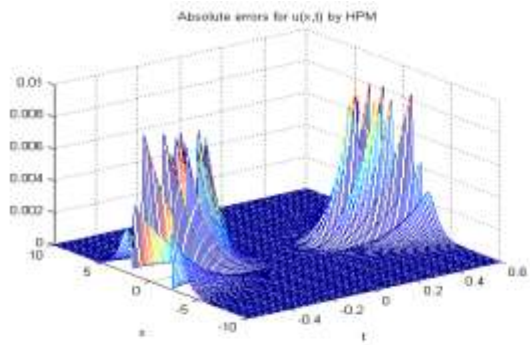


Figure (8): Absolute errors between Exact solution and HPM for $u(x,t)$.

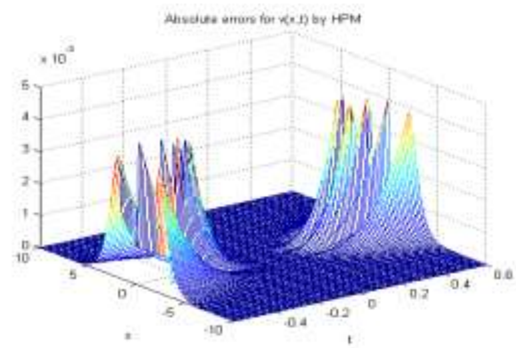


Figure (9): Absolute errors between Exact solution and HPM for $v(x,t)$.

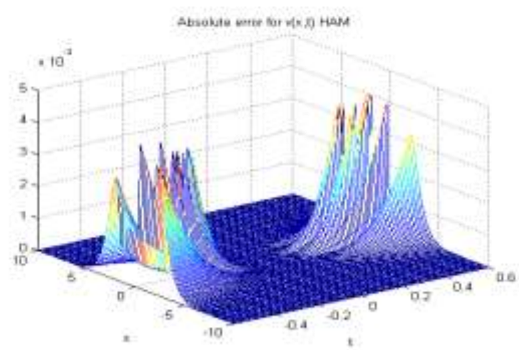


Figure (10): Absolute errors between Exact solution and HAM for $v(x,t)$.

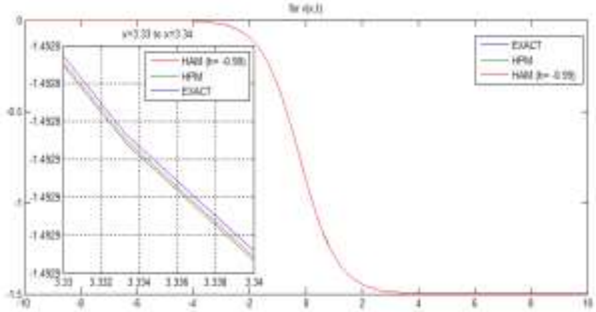


Figure (11): Zooming curves for Exact, HPM and (HAM$^{\hbar = -0.99}$)for $v(x,t)$ when $x \in (3.33, 3.34)$, $w = 1.5$ and $t = -0.1515$.

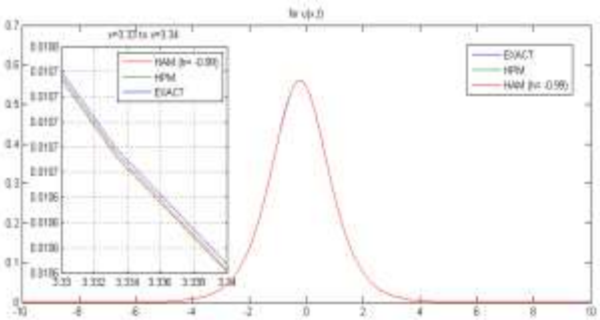


Figure (12): Zooming curves for Exact, HPM and (HAM$^{\hbar = -0.99}$)for $u(x,t)$ when $x \in (3.33, 3.34)$, $w = 1.5$ and $t = -0.1515$.

Table (1): Absolute errors between exact and approximation solutions by HPM and ( HAM $^{\hbar = -0.99}$) when $x \in (-10, 10)$ and $t = -0.1515$ for $u(x,t)$.

| t | X | EXACT | HPM | HAM ($\hbar = -1$) | HAM ($\hbar = -0.99$) | \|EXACT-HPM\| | \|EXACT-HAM($\hbar = -0.99$)\| |
|---|---|---|---|---|---|---|---|
| -0.1515 | -5.96 | 4.1471320E-04 | 4.1453597E-04 | 4.1453597E-04 | 4.1447303E-04 | 1.7722733E-07 | 2.4017132E-07 |
| | -5.76 | 5.6142994E-04 | 5.6119037E-04 | 5.6119037E-04 | 5.6110520E-04 | 2.3957377E-07 | 3.2473646E-07 |
| | -5.56 | 7.6001686E-04 | 7.5969319E-04 | 7.5969319E-04 | 7.5957800E-04 | 3.2366865E-07 | 4.3886397E-07 |
| | -5.35 | 1.0287829E-03 | 1.0283460E-03 | 1.0283460E-03 | 1.0281902E-03 | 4.3694472E-07 | 5.9271041E-07 |
| | -5.15 | 1.3924753E-03 | 1.3918861E-03 | 1.3918861E-03 | 1.3916755E-03 | 5.8924601E-07 | 7.9977288E-07 |
| | -4.95 | 1.8845233E-03 | 1.8837298E-03 | 1.8837298E-03 | 1.8834455E-03 | 7.9350036E-07 | 1.0778616E-06 |
| | -4.75 | 2.5500473E-03 | 2.5489808E-03 | 2.5489808E-03 | 2.5485970E-03 | 1.0664823E-06 | 1.4502455E-06 |
| | 3.33 | 1.0678328E-02 | 1.0671885E-02 | 1.0671885E-02 | 1.0674305E-02 | 6.4427918E-06 | 4.0229485E-06 |
| | 3.54 | 7.9064651E-03 | 7.9014201E-03 | 7.9014201E-03 | 7.9032630E-03 | 5.0450726E-06 | 3.2021110E-06 |
| | 3.74 | 5.8503127E-03 | 5.8464267E-03 | 5.8464267E-03 | 5.8478188E-03 | 3.8859996E-06 | 2.4939512E-06 |
| | 3.94 | 4.3268003E-03 | 4.3238416E-03 | 4.3238416E-03 | 4.3248868E-03 | 2.9587263E-06 | 1.9135495E-06 |
| | 4.14 | 3.1988957E-03 | 3.1966615E-03 | 3.1966615E-03 | 3.1974428E-03 | 2.2341710E-06 | 1.4528510E-06 |
| | 4.34 | 2.3643896E-03 | 2.3627126E-03 | 2.3627126E-03 | 2.3632948E-03 | 1.6770323E-06 | 1.0948199E-06 |
| | 4.55 | 1.7472440E-03 | 1.7459906E-03 | 1.7459906E-03 | 1.7464234E-03 | 1.2534010E-06 | 8.2057140E-07 |
| | 5.96 | 2.0975732E-04 | 2.0960261E-04 | 2.0960261E-04 | 2.0965535E-04 | 1.5471545E-07 | 1.0197829E-07 |
| | 6.16 | 1.5492943E-04 | 1.5481504E-04 | 1.5481504E-04 | 1.5485401E-04 | 1.1438710E-07 | 7.5414275E-08 |
| | 6.36 | 1.1443139E-04 | 1.1434684E-04 | 1.1434684E-04 | 1.1437564E-04 | 8.4547975E-08 | 5.5751338E-08 |
| | 6.57 | 8.4518606E-05 | 8.4456126E-05 | 8.4456126E-05 | 8.4477401E-05 | 6.2480266E-08 | 4.1205076E-08 |
| | 6.77 | 6.2424701E-05 | 6.2378535E-05 | 6.2378535E-05 | 6.2394252E-05 | 4.6165622E-08 | 3.0448625E-08 |
| | 6.97 | 4.6106099E-05 | 4.6071992E-05 | 4.6071992E-05 | 4.6083602E-05 | 3.4107302E-08 | 2.2497111E-08 |
| | 7.17 | 3.4053256E-05 | 3.4028060E-05 | 3.4028060E-05 | 3.4036636E-05 | 2.5196560E-08 | 1.6620461E-08 |
| Mean square error | | | | | | 3.9674073E-09 | 3.8860799E-09 |

Table (2): Absolute errors between exact and approximation solutions by HPM and ( HAM $\hbar = -0.99$) when $x \in (-10,10)$ and $t = -0.1515$ for $v(x,t)$.

| t | x | EXACT | HPM | HAM ($\hbar = -1$) | HAM ($\hbar = -0.99$) | \|EXACT-HPM\| | \|EXACT-HAM$_{\hbar = -0.99}$\| |
|---|---|---|---|---|---|---|---|
| -0.1515 | -5.96 | -2.7652644E-04 | -2.7640802E-04 | -2.7640802E-04 | -2.7636602E-04 | 1.1841903E-07 | 1.6042392E-07 |
| | -5.76 | -3.7438007E-04 | -3.7421986E-04 | -3.7421986E-04 | -3.7416301E-04 | 1.6020595E-07 | 2.1705846E-07 |
| | -5.56 | -5.0684917E-04 | -5.0663249E-04 | -5.0663249E-04 | -5.0655556E-04 | 2.1667693E-07 | 2.9361562E-07 |
| | -5.35 | -6.8616915E-04 | -6.8587621E-04 | -6.8587621E-04 | -6.8577211E-04 | 2.9294084E-07 | 3.9704446E-07 |
| | -5.15 | -9.2889209E-04 | -9.2849625E-04 | -9.2849625E-04 | -9.2835542E-04 | 3.9584131E-07 | 5.3666851E-07 |
| | -4.95 | -1.2574029E-03 | -1.2568684E-03 | -1.2568684E-03 | -1.2566780E-03 | 5.3450993E-07 | 7.2495571E-07 |
| | -4.75 | -1.7019626E-03 | -1.7012416E-03 | -1.7012416E-03 | -1.7009841E-03 | 7.2106524E-07 | 9.7850293E-07 |
| | 3.33 | -1.4928470E+00 | -1.4928518E+00 | -1.4928518E+00 | -1.4928501E+00 | 4.7999078E-06 | 3.0891058E-06 |
| | 3.54 | -1.4947104E+00 | -1.4947140E+00 | -1.4947140E+00 | -1.4947127E+00 | 3.6433760E-06 | 2.3608641E-06 |
| | 3.74 | -1.4960896E+00 | -1.4960923E+00 | -1.4960923E+00 | -1.4960914E+00 | 2.7453116E-06 | 1.7876303E-06 |
| | 3.94 | -1.4971099E+00 | -1.4971120E+00 | -1.4971120E+00 | -1.4971112E+00 | 2.0576197E-06 | 1.3445550E-06 |
| | 4.14 | -1.4978644E+00 | -1.4978659E+00 | -1.4978659E+00 | -1.4978654E+00 | 1.5362038E-06 | 1.0063998E-06 |
| | 4.34 | -1.4984221E+00 | -1.4984232E+00 | -1.4984232E+00 | -1.4984228E+00 | 1.1436548E-06 | 7.5062768E-07 |
| | 4.55 | -1.4988343E+00 | -1.4988351E+00 | -1.4988351E+00 | -1.4988348E+00 | 8.4963522E-07 | 5.5841002E-07 |
| | 5.96 | -1.4998601E+00 | -1.4998603E+00 | -1.4998603E+00 | -1.4998602E+00 | 1.0334720E-07 | 6.8150430E-08 |
| | 6.16 | -1.4998967E+00 | -1.4998968E+00 | -1.4998968E+00 | -1.4998968E+00 | 7.6369150E-08 | 5.0366170E-08 |
| | 6.36 | -1.4999237E+00 | -1.4999238E+00 | -1.4999238E+00 | -1.4999237E+00 | 5.6425930E-08 | 3.7216660E-08 |
| | 6.57 | -1.4999437E+00 | -1.4999437E+00 | -1.4999437E+00 | -1.4999437E+00 | 4.1686580E-08 | 2.7496840E-08 |
| | 6.77 | -1.4999584E+00 | -1.4999584E+00 | -1.4999584E+00 | -1.4999584E+00 | 3.0795130E-08 | 2.0313700E-08 |
| | 6.97 | -1.4999693E+00 | -1.4999693E+00 | -1.4999693E+00 | -1.4999693E+00 | 2.2748040E-08 | 1.5006050E-08 |
| | 7.17 | -1.4999773E+00 | -1.4999773E+00 | -1.4999773E+00 | -1.4999773E+00 | 1.6803080E-08 | 1.1084660E-08 |
| Mean square error | | | | | | 7.7841157E-10 | 7.7558104E-10 |

## IV. CONCLUSION

In this paper, The HPM and HAM, which are two powerful numerical methods, have been used for solving Kaup-Boussinesq system. By applying these methods on the Kaup-Boussinesq system we have got the required results. The results were convenient and showed that both methods were suitable and effective for solving such kind of problems. But HAM appeared to be more accurate and efficient than HPM because it contains auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region of solution series. And when $\hbar = -1$ showed that the series of HPM is exactly as HAM, as it appear in table (1, 2), that is why HPM is a special case of HAM.

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