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Two Problems in Integral Sum Labeling

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Abstract- Harary [4,5] introduced the concepts of sum graph (∑-graph) and integral sum graph (∫∑-graph). In this paper, we prove that graphs K_1□ (K_1□ ∪ □ K_3) and K_1□ (K_2 ∪ K_3) are not ∫∑-graphs though they are spanning sub-graph of integral sum graphs $G_{1,3}$ and $G_{2,3}$, respectively. A graph G is an integral sum graph or ∫∑-graph if the vertices of G can be labeled with distinct integers so that e = uv is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G.
Keywords – sum graph or ∑-graph, integral sum graph or ∫∑-graph, maximal sum graph G_n, integral sum graph $G_{\Delta n}$, integral sum graph $G_{m,n}$, comparable and non-comparable graphs, semi-group.

I. INTRODUCTION

The concept of sum graphs and integral sum graphs are introduced by Harary [4,5]. Harary [5] introduced a family of integral sum graphs $G_{n,n} = K_1+(G_n+G_n)$ over a set of integers {-n,…,-2,-1, 0,1,2,…,n}. Vilfred [9,10] defined $G_{m,n}$, a generalized graph of $G_{n,n}$ and derived different properties of it. Vilfred proved that $G_{m,n}$ is a maximal integral sum graph of order m+n+1 for m,n∈N ∪ {0}□ □ and the maximal integral sum graphs of order 2n and 2n+1 are $G_{n-1,n}$ and $G_{n,n}$, respectively, n∈N Though the graph $G_{m,n}$ is an integral sum graph for m,n∈N ∪ {0}□, all spanning sub graphs of $G_{m,n}$ need not be integral sum graphs. Establishing a given graph as not an integral sum graph is a difficult problem. In this paper we prove that the graphs K_1□ (K_1□ ∪ □ K_3) and K_1□ (K_2 ∪ K_3) are not integral sum graphs though they are spanning sub-graph of integral sum graphs $G_{1,3}$ and $G_{2,3}$, respectively. See figures 1 and 2. For all basic ideas in graph theory, we follow [3]. Now let us consider some basic definitions and results on sum and integral sum graphs.

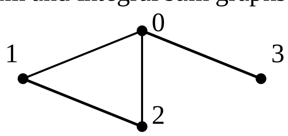

Fig.1. $K_1+(K_1 \cup K_2)$

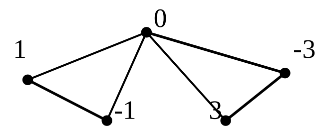

Fig.2. $K_1+(K_2 \cup K_2)$

Definition 1.1 [5] A graph G is an integral sum graph or ∫∑-graph if the vertices of G can be labeled with distinct integers so that uv is an edge of G if and only if the sum of the labels on vertices u and v is also a label in V(G). If G is an integral sum graph with respect to a label set S, then G can be denoted as $G = G^+(S)$.

Chen [1] obtained several properties of the integral sum labeling of a graph G with $\Delta(G) < |V(G)|-1$. Nicholas, Somasundaram and Vilfred [6] studied the general properties of connected integral sum graph G with $\Delta(G) = |V(G)|-1$. Vilfred [9,10] defined and studied maximal integral sum graphs. We give below some important properties of integral sum graphs.

Definition 1.2 Let G be a connected graph with maximum degree $\Delta(G) = |V(G)|-1$. Define $V_\Delta(G) = \{x \in V(G) / deg(x) = \Delta(G)\}$.

Theorem 1.3 [1] Let f be an ∫∑-labeling of a non-trivial graph G. Then $f(x) \neq 0$ for every vertex x of G if and only if $\Delta(G) < |V(G)|-1$. ■

Theorem 1.4 [6] Let f be an ∫∑-labeling of a connected graph G and x,u,v,w∈V(G). If $f(u) \neq -f(x)$, then for each v∈□ (u) such that f(u)+f(v) = f(w), either w = x or xw∈E(G). ■

Theorem 1.5 [6] Let f be an ∫∑-labeling of a connected graph G with $\Delta(G) = |V(G)|-1$. If $|V_\Delta(G)| \geq 2$, then

(i) There exists a vertex $x \in V_\Delta(G)$ such that f(x) = 0 and

(ii) For every vertex $y \in V_\Delta(G) \setminus \{x\}$, there exists a vertex $y' \in$□ $V(G) \setminus V_\Delta(G)$ such that f(y)+f(y') = 0. ■

Theorem 1.6 [6] Let f be an ∫∑-labeling of a connected graph G of order k+2 with $\Delta(G) = |V(G)|-1$ and $|V_\Delta(G)| \geq 2$. If $y \in V_\Delta(G)$ such that $f(y) \neq 0$, then for every vertex u∈V(G), f(u) = r.f(y) where r∈{0,1,-1,-2,...,-k}, k∈N. ■

Theorem 1.7 [6] If G ($\neq K_3$) is a connected ∫∑-graph with $\Delta(G) = |V(G)|-1$, then $|V_\Delta(G)| \leq 2$. ■

Theorem 1.8 [6] Integral sum graphs G ($\neq K_3$) of order n with $|V_\Delta(G)| = 2$ are unique up to isomorphism. (This graph of order n is denoted by $G_{\Delta n}$.) ■

Definition 1.9 [10] Let $G_{m,n} = G^+(S)$ where S = {-m,-m+1,...,-2,-1,0,1,2,...,n}, m,n∈ m,n∈N ∪ {0}□ . Clearly, $G_{m,n} = K_1+(G_m+G_n) = G_{n,m}$ is an integral sum graph, m,n∈□ m,n∈N ∪ {0}□ Without loss of generality, hereafter, we consider m ≤ n for the graph $G_{m,n}$. Integral sum graphs $G_{4,4}$ and $G_{4,5}$ are given in figures 3 and 4.

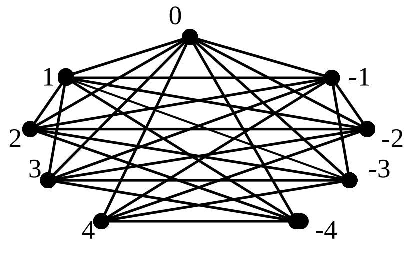

Fig.3. $G_{4,4}$

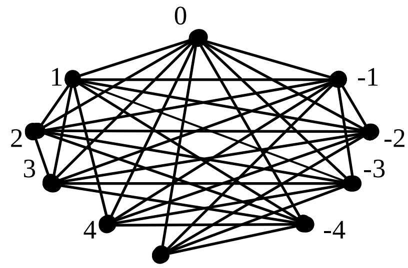

Fig.4. $G_{4,5}$

Theorem 1.10 [10] $G_{m,n}$ is an integral sum graph for m,n∈□ m,n∈N ∪ {0}□ and is a maximal integral sum graph for m,n∈□N.□ □ The maximal integral sum graphs of order 2n and 2n+1 are $G_{n-1,n}$ and $G_{n,n}$, respectively, n∈N ■

For further readings on graph labeling problems refer [2].

II. MAIN RESULTS

Graph $G_{m,n} = K_1+(G_m+G_n)$ is an integral sum graph form, m,n∈N ∪ {0}□ ,[10]. An interesting problem is given two graphs, G and H, whether K_1+(G +□ H) is an integral sum graph or not? A special case of the above problem is the case in which G and H are either sum or integral sum graphs. The answer to the above problem seems to be difficult. Graphs $K_1+(K_1$ +□ $K_2)$ and $K_1+(K_2 + K_2)$ are ∫Σ-graphs, see figures 1 and 2., whereas $K_1+(K_1 + K_3)$ and $K_1+(K_2$ +□ $K_3)$ are not ∫Σ-graphs, see problems 1 and 2. Clearly the two graphs are spanning sub-graph of integral sum graphs $G_{1,3}$ and $G_{2,3}$, respectively.

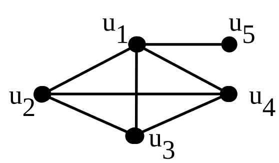

Fig.5. $G = K_1+(K_1 \cup K_3)$

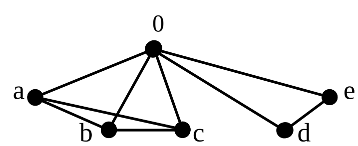

Fig.6. $G = K_1+(K_2 \cup K_3)$

Problem 2.1 The graph $K_1+(K_1 \cup K_3)$ is not an ∫Σ–graph.

Solution

Let $G = K_1+(K_1 \cup K_3)$ and V(G) = $\{u_1,u_2,u_3,u_4,u_5\}$. If possible, let f be an ∫Σ□ labeling of G, then using theorem 1.3, $f(u_1) = 0$. Let $f(u_2) = a$, $f(u_3) = b$, $f(u_4) = c$ and $f(u_5) = d$ where 0,a,b,c,d are all distinct integers. See figure 5. Therefore, {0, a, b, c, d} = {0, a, b, c, d, a+b, b+c, c+a} where a+b, b+c and c+a are all different. This implies, two of them are neither equal nor equal to 0 or d. This implies, the three values are 0, d and the third being one of a,b,c in some order. Let a+b = 0, b+c = d and (then) a+c =b. All other cases are similar to the above case only.

This implies, b = -a = a+c which implies, c = -2a. This implies, d = -3a which implies, a+d = -2a = c. This implies, a and d are adjacent in G which is a contradiction. Hence the graph $K_1+(K_1 \cup K_3)$is not an ∫Σ–graph. ■

Problem 2.2 The graph $G = K_1+(K_2 \cup K_3)$ is not an ∫Σ□ graph.

Solution Let $G = K_1+(K_2 + K_3)$ and V(G) = $\{u_1, u_2, u_3, u_4, u_5, u_6\}$. If possible, let f be an ∫Σ□ labeling of G. Then using theorem 1.3, $f(u_1) = 0$. Let $f(u_2) = a$, $f(u_3) = b$, $f(u_4) = c$, $f(u_5) = d$ and $f(u_6) = e$ where 0,a,b,c,d,e are all distinct integers. See figure 6.Therefore,{0, a, b, c, d, e} = {0, a, b, c, d, e, a+b, b+c, c+a, d+e} where a+b, b+c and a+c are all different. This implies, two of them are neither equal nor equal to 0, d or e.

Now let us consider the following possible cases and their sub-cases. All other cases are similar to the above cases only.

1. a+b = 0 = d+e and b+c, c+a $\neq$ 0 (Two among a+b, b+c, c+a, d+e are 0.).
1.1 a+b = 0 = d+e; b+c = a; c+a = d (or e). In this case, we get, b = -a; c = 2a; d = 3a and b+d = 2a = c which implies, u_3 and u_5 are adjacent in G which is a contradiction. Thus this case is not possible.
1.2 a+b = 0 = d+e; b+c = d; c+a = e.In this case, we get, 0 = d+e = (b+c)+(c+a) = 2c whic implies, c = 0, a contradiction. Hence this case is not possible.
2. a+b = 0; b+c, c+a, d+e $\neq$ 0 (One among a+b, b+c, c+a is 0 and d+e $\neq$ 0.).
Different possible sub-cases are as follows.
2.1. a+b = 0; d+e = a; b+c, c+a $\neq$ 0.
2.1.1. a+b = 0; d+e = a; b+c = a; c+a = d (or e).
In this case we get, c = a-b = 2a; d = c+a = 3a; e = a-d = -2a which implies, c+e = 0. This implies, u_4 and u_6 are adjacent in G which is a contradiction. Hence this case is not possible.
2.1.2. a+b = 0; d+e = a; b+c = d(or e) ; c+a = b.
In this case we get, c = b-a = -2a; d = b+c = -3a. ; e=4a. This implies, a+d = -2a = c. This implies, u_2 and u_5 are adjacent in G which is a contradiction. Hence this case is not possible.
2.1.3. a+b = 0; d+e = a; b+c = e; (or d) and c+a = d. (or e).
In this case, a = d+e = (c+a)+(b+c) = 2c; d = c+a = 3c;b = -a = -2c which implies, b+d = c. This implies u_3 and u_5 are adjacent in G which is a contradiction. Hence this case is not possible.
2.2. a+b = 0; d+e = c; b+c, c+a $\neq$ 0.
2.2.1. a+b = 0; d+e = c; b+c = d; b+c = e; (or d) and c+a = d. (or e).
In this case we get, c = d+e = (b+c)+(c+a) = 2c. This implies, c = 0 which is a contradiction. Hence this case is not possible.
2.2.2. a+b = 0; d+e = c; b+c = a; c+a = d. (or e).
In this case, b = -a; c = a-b = 2a; d = c+a = 3a; e = c-d = -a. This implies, a+e = 0 which implies u_2 and u_6 are adjacent in G. This is a contradiction and hence this case is not possible.
3. d+e = 0 and a+b, b+c, c+a $\neq$ 0 (None of a+b, b+c, c+a is 0 and d+e = 0.).
In this case, a+b, b+c, c+a take at the most one value among a, b, c and two among the three, a+b, b+c, c+a take values d and e. And so let a+b = c; b+c = d and c+a = e. This implies, 0 = d+e = (b+c)+(c+a) = 3c. This implies, c = 0 which is a contradiction. Hence this case is not possible.
4. a+b, b+c, c+a, d+e $\neq$ 0 (None of a+b, b+c, c+a, d+e is 0.).
4.1. d+e = a; a+b = d; b+c = a; c+a = e.
In this case, a = d+e = (a+b)+(c+a) = 3a. This implies, a = 0 which is a contradiction. Hence this case is not possible.
4.2. d+e = a; a+b = c; b+c = e; c+a = d.
In this case, a = d+e = (c+a)+(b+c) = 3c; b = c-a = -2c and e = b+c = -c. This implies, c+e = 0 which implies, u_4 and u_6 are adjacent in G. This is a contradiction. Hence this case is not possible.
Thus in all possible cases we get a contradiction and hence $\int\Sigma$ labeling doesn't exist for the graph G = $K_1+(K_2 \cup K_3)$. Hence the graph $K_1+(K_2 \cup K_3)$ is not an $\int\Sigma$ graph. ■

Remark 2.3 $G_{k,m-k}$ are the only maximal $\int\Sigma$-graphs of order m+1 having at least one of it's vertices is of degree m, m,k+1,m-k+1$\in$N. Though any proper spanning super graph of a maximal $\int\Sigma$-graph need not be an $\int\Sigma$-graph, the converse part is not known. For example, $K_1+(K_1 \cup K_3)$ and $K_1+(K_2 \cup K_3)$ are not $\int\Sigma$-graphs. Secondly whether $G_{k,m-k}$'s are the only maximal $\int\Sigma$-graphs of order m+1, m,k+1,m-k+1$\in$N. Thirdly when the graph $K_1+(G \cup H)$ is an $\int\Sigma$-graph?

Remark 2.4 Let u be a vertex of degree n-1 in an $\int\Sigma$-graph G of order n $\geq$ 3. Clearly, G $\neq$ K_n for n $\geq$ 3. Then by adding a new edge e, by joining two non-adjacent vertices of G, to G and joining a new vertex v with u, we may think that the resultant graph H is also an $\int\Sigma$-graph. But it is not always the case. The graph G = $K_{1,2}$ = K_3-e and its new graph H = $K_1+(K_1 \cup K_2)$ are $\int\Sigma$-graphs whereas G = K_4-e is an $\int\Sigma$-graph but its new graph H = $K_1+(K_1 \cup K_3)$ is not an $\int\Sigma$-graph. See figures 7,8,9 and 10 and problem 2.1. But the general result is not known.

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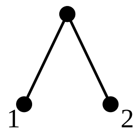

Fig.7. G = K_3-e

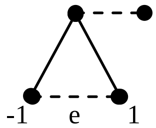

Fig.8. H = $K_1+(K_1 \cup K_2)$

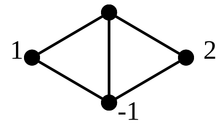

Fig.9. G = K_4-e

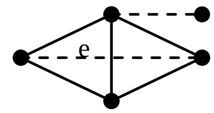

Fig.10. H = $K_1+(K_1 \cup K_3)$

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