Two special Dio-quadruples generated through Euler Polynomials

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Abstract- This paper has two sections A and B. Section A deals with the study of formulating special Dio-quadruples (a,b,c,d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by two is a perfect square. Section B concerns with constructing special Dio- quadruples (a,b,c,d) generated through Euler polynomials such that the product of any two of the set minus their sum and increased by five is a perfect square.

Keywords – Dio-quadruples, Pell equation.

I. INTRODUCTION

The problem of constructing the sets with property that the product of any two of its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus [1].A set of m positive integers $\{a_1, a_2, ...a_m\}$ is said to have the property $D(n), n \in z - \{0\}$ if $a_i a_j + n$, a perfect square for all $1 \le i \le j \le m$ and such a set is called a Diophantine m-tuples with property $D(n)$. Many mathematicians considered the construction of different formulations of Diophantine quadruples with the property $D(n)$ for any arbitrary integer n and also for any linear polynomials in n. This paper aims at constructing special Dio-quadruples where the special mention is provided because it differs from the earlier one and the special Dio-quadruples is constructed where the product of any two members of the quadruple with the addition or subtraction of members and increased by a non-zero integer or a polynomial with integer coefficients satisfies the required property [2-4]. In particular, given two Euler polynomials, this paper searches for four distinct polynomials with integer coefficients such that the product of any two polynomials minus their sum and increased by two and five respectively is a perfect square.

II. SECTION A : FORMULATION OF $D(2)$ DIO-QUADRUPLES

Consider the Euler polynomials $E_1(x)$ and $E_2(x)$ given by

$E_1(x) = x - \frac{1}{2}$, $E_2(x) = x^2 - x$

Let $a = [2E_1(x)]^2 = 4x^2 - 4x + 1$ and $b = E_2(x) = x^2 - x$ be two polynomials such that $ab - (a+b) + 2$ is a perfect square.

Let $c_N(x)$ be any non-zero polynomial such that

$$(a-1)c_N(x) - a + 2 = p_N^2 \tag{1}$$

$$(b-1)c_N(x) - b + 2 = q_N^2 \tag{2}$$

Eliminating $c_N(x)$ from (1) and (2), we get

$$(b-1)p_N^2 - (a-1)q_N^2 = (b-a) \tag{3}$$

Setting

$$p_N = X_N + (a-1)T_N \tag{4}$$

$$q_N = X_N + (b-1)T_N \tag{5}$$

in (3), we have

$$X_N^2 = (b-1)(a-1)T_N^2 + 1 \tag{6}$$

whose initial solution is $T_0 = 1$, $X_0 = 2x^2 - 2x - 1$

The general solution of (6) is

$$X_N = \frac{1}{2} f_N, \quad T_N = \frac{1}{2\sqrt{(a-1)(b-1)}} g_N \tag{7}$$

where

$$f_N = \left(X_0 + \sqrt{(a-1)(b-1)}T_0\right)^{N+1} + \left(X_0 - \sqrt{(a-1)(b-1)}T_0\right)^{N+1}$$

$$g_N = \left(X_0 + \sqrt{(a-1)(b-1)}T_0\right)^{N+1} - \left(X_0 - \sqrt{(a-1)(b-1)}T_0\right)^{N+1}$$

Substitution of (7) in (4) gives

$$p_N = \frac{1}{2} f_N + (a-1) * \frac{1}{2\sqrt{(a-1)(b-1)}} g_N \tag{8}$$

From (8) and (1) we get

$$c_N(x) = 1 + \frac{(p_N^2 - 1)}{(a-1)} \tag{9}$$

Substituting $N = 0$ in (9) we have,

$$c_0(x) = 1 + \frac{(p_0^2 - 1)}{(a-1)}$$

(i.e) $c_0(x) = 9(x^2 - x) - 2$

Note that the tuple $(a, b, c_0(x))$ is Dio-triple with property $D(2)$.

Again, substituting $N = 1,2$ in (9) and simplifying we get,

$$c_1(x) = 144[E_2(x)]^3 - 192[E_2(x)]^2 + 76E_2(x) - 7$$

$$c_2(x) = 1 + \{48[E_2(x)]^2 - 56E_2(x) + 15\}\{48[E_2(x)]^3 - 56[E_2(x)]^2 + 15E_2(x) - 1\}$$

It is seen that $(a, b, c_0(x), c_1(x))$ and $(a, b, c_1(x), c_2(x))$ represent Dio-quadruples with property $D(2)$ respectively.

In general, it is observed that the quadruple $(a, b, c_{N-1}(x), c_N(x)), N = 1,2,3,\ldots\ldots$ is a Dio-quadruple with property $D(2)$. Some numerical examples are given in Table 1 below:

Table 1: $\boldsymbol{D(2)}$ Dio-quadruples

$\boldsymbol{x}$	$\boldsymbol{(a, b, c_0(x), c_1(x))}$	$\boldsymbol{(a, b, c_1(x), c_2(x))}$	$\boldsymbol{(a, b, c_2(x), c_3(x))}$
2	(9,2,16,529)	(9,2,529,17956)	(9,2,17956,609961)
3	(25,6,52,24641)	(25,6,24641,11876488)	(25,6,11876488,5724442153)

III. SECTION B: FORMULATION OF $\boldsymbol{D(5)}$ DIO-QUADRUPLES

Consider the Euler polynomials $E_0(x)$ and $E_1(x)$ given by

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}$$

Let $a = E_0(x) = 1$ and $b = [2E_1(x)] = 2x - 1$ be two polynomials such that $ab - (a+b) + 5$ is a perfect square.

Let c be any polynomial. Observe that, $ac - (a+c) + 5$ is automatically a perfect square.

Now, consider

$$bc - (b+c) + 5 = p^2 \tag{10}$$

After some calculations, it is seen that (10) is satisfied when $c = 2x + 3$.
Note that the triple $(1,\ 2x-1,\ 2x+3)$ is a Dio-triple with property $D(5)$.
Let d be any non-zero polynomial such that

$$bd - (b+d) + 5 = \alpha^2 \tag{11}$$

$$cd - (c+d) + 5 = \beta^2 \tag{12}$$

Eliminating d between the above two equations we have,

$$(2x+2)\alpha^2 - (2x-2)\beta^2 = 16 \tag{13}$$

Taking

$$\alpha = X + (2x-2)T \tag{14}$$

$$\beta = X + (2x+2)T \tag{15}$$

in (13), we get

$$X^2 = (4x^2 - 4)T^2 + 4$$

which is satisfied by

$$T = 1,\ X = 2x \tag{16}$$

Using (16) in (14) and in view of (11), one obtains

$$d = 8x + 1$$

Observe that $(1, 2x-1,\ 2x+3,\ 8x+1)$ is a Dio-quadruple with property $D(5)$.
The repetition of the above process leads to the generation of Dio-quadruples given by $(1,\ 2x+3,\ 8x+1, 18x+7)$, $(1, 8x+1, 18x+7, 50x+11)$, $(1, 18x+7, 50x+11, 128x+33)$ and so on.

IV. CONCLUSION

In this paper, we have illustrated the process of obtaining special Dio-quadruples with properties $D(2)$ and $D(5)$ generated through Euler polynomials $E_1(x), E_2(x)$ and $E_0(x), E_1(x)$. To conclude, one may search for special Dio-quadruples with suitable properties by employing Euler polynomials.

V. REFERENCE

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