Order Reduction Using Modified Cauer Form and Pade Approximation Method

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Abstract- In this paper, the author presents a mixed method for reducing the order of large-scale dynamic systems by combining the Modified Cauer Form and Padé Approximation. In the proposed approach, the denominator coefficients of the reduced-order model are obtained using the Modified Cauer Form, while the numerator coefficients are derived using the Padé Approximation. This hybrid technique is conceptually straightforward and consistently produces stable reduced-order models, provided the original high-order system is stable. The effectiveness of the proposed method is demonstrated through numerical examples drawn from existing literature.

Keywords – Modified Cauer form, Order reduction, Pade approximation, Stability, Transfer function.

I. INTRODUCTION

The analysis of high-order systems is often both tedious and computationally expensive, as such systems tend to be too complex for practical applications. To address this challenge, mathematical techniques are commonly employed to develop simplified models that accurately approximate the behavior of the original high-order systems. Model order reduction plays a crucial role in the analysis, synthesis, and simulation of practical systems. Numerous numerical methods have been proposed in the literature for the reduction of large-scale systems. A wide variety of model reduction techniques [1–5] have been developed in the frequency domain by several researchers. However, none of these methods consistently yield optimal results across all scenarios.

Among the available techniques, the Modified Cauer Form [6] is one of the most popular. It offers the advantage of avoiding the computation of time moments and Markov parameters of the original system. Moreover, it guarantees the stability of the reduced-order model, provided the original system is stable. The Padé Approximation method, originally introduced by Padé [9], is computationally straightforward and effectively matches initial time moments and steady-state values. However, a key limitation of the Padé method is that it may produce an unstable reduced model, even when the original system is stable.

To address this issue, Shamash [3] proposed a reduction method that retains the poles of the high-order system in the reduced model and applies Padé approximation around multiple expansion points. In the present work, a mixed approach is proposed wherein the denominator polynomial of the reduced-order model is determined using the Modified Cauer Form, while the numerator coefficients are obtained via Padé Approximation. The proposed method is evaluated and compared with several established order reduction techniques from the literature.

The remainder of the paper is structured as follows: the proposed method is described in Section III, numerical results are presented in Section V, and concluding remarks are provided in Section VI.

II. STATEMENT OF THE PROBLEM

Assume the transfer function of high order original system of the order ‘n^{th}’ and its mth order reduced order model respectively are

$$G_n(S) = \frac{c_{1,1} + c_{1,2}\, s + c_{1,3} s^2 + \cdots \ldots\ldots\ldots\ldots + c_{1,n} s^{n-1}}{d_{1,1} + d_{1,2} s + d_{1,3} s^2 + \cdots \ldots\ldots\ldots\ldots + d_{1,n} s^{n-1} + s^n} \quad (1)$$

Where c_{1i} & d_{1i}, $1 \leq i \leq$ n are known scalar constants

$$R_m(S) = \frac{N_m(s)}{D_m(s)} = \frac{c_{m,1} + c_{m,2}\, s + c_{m,3} s^2 + \cdots \ldots\ldots\ldots + c_{m,m} s^{m-1}}{d_{m+1,1} + d_{m+1,2} s + d_{m+1,3} s^2 + \cdots \ldots\ldots\ldots + d_{m+1,m} s^{m-1} + s^m} \quad (2)$$

Where $c_{m,j}$ & $d_{m+1,j}$, $1 \leq j \leq m$ are known scalar constants

The aim of order reduction is to realize the mth order reduced model as in equation (2) from the original system as in equation (1) such that it preserves the important properties of the original high order system.

III. PROPOSED METHOD

The **denominator polynomial** of the reduced mth order model is determined using the **Modified Cauer Form**.

Step-1: To obtain of the denominator polynomial $D_m(s)$ for the m th order reduced model by using modified Cauer form: The procedure for getting $D_m(s)$ using modified Cauer form is as follows: Consider the given higher order system $G_n(s)$ Without loss of generality, the coefficient of highest power of sn in equations (1) & (2) can always made unity and numerator is one degree lower than the denominator. $G_n(s)$ can be expanded into a Cauer type continued fraction about s=0 and s=∞ alternately as follows

$$Gn(s) = \cfrac{1}{\frac{d\,1{,}1}{c\,1{,}1} + \cfrac{s}{\frac{c1{,}n}{d2{,}n} + \cfrac{1}{\frac{d2{,}1}{c2{,}1} + \cdots \ldots\ldots\ldots + \cfrac{s}{\frac{c2{,}n-1}{d3-1} + \cfrac{1}{\ldots\ldots}}}}}$$

Where

$d_{2,1} = d_{1,2} - \frac{d1{,}1C1{,}2}{C1{,}1}$,　　　　$c_{2,1} = c_{1,1} - \frac{c1{,}nC2{,}1}{C2{,}n}$,

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$d_{2,n-1} = d_{1,n} - \frac{d1{,}1C1{,}n}{C1{,}1}$,　　　　$c_{2,n-1} = c_{1,n-1} - \frac{c1{,}nd2{,}n-1}{d2{,}n}$,

$d_{2,n} = 1$

By following the above sequence of expansion we get the following form

$$G_n(s) \cfrac{1}{h_1 + \cfrac{s}{H_1 + \cfrac{1}{h_2 + \cfrac{s}{H_2 + \cfrac{1}{\ldots\ldots}}}}} \quad (3)$$

Where the quotients h_1, h_2…………………… H_2, H_1 Calculated from the following modified array

$d_{1,1}$	$d_{1,2}$	$d_{1,n-1}$	$d_{1,n}$	1
$c_{1,1}$	$c_{1,2}$	$c_{1,n-1}$	$c_{1,n}$	
$d_{2,1}$	$d_{2,2}$	$d_{2,n-1}$	1	
$c_{2,1}$	$c_{2,2}$	$c_{2,n-1}$		
$d_{3,1}$	$d_{3,2}$	1		
……………………	……………………	……………………	……………………	
……………………	……………………	……………………	……………………	
$C_{n-1,1}$	$c_{n-1,2}$			
$D_{n,1}$				
$C_{n,1}$				
1				

$h_1 = \frac{d_{1,1}}{c_{1,1}}$,　$h_2 = \frac{d_{2,1}}{c_{2,1}}$, ………………………………………………… $h_n = = \frac{d_{n,1}}{c_{n,1}}$

$H_1 = c_{1n}$ $\Big\}$

$H_2 = c_{2,n-1}$ (4)
$H_{n-1} = c_{n-1,2}$
$H_n = c_{n,1}$

The denominator and numerator coefficients of equation (1) form the first and second rows and remaining elements starting with third row onwards are evaluated from the recursive relations.

$D_{j+1,k} = d_{j,k+1} - h_j c_{j,k+1}$ and $c_{j+1,k} = c_{j,k} - H_j d_{j+1,k}$ j= 1,2,3…….. n-1

Here $h_j = \frac{d_{j,1}}{c_{j,1}}$ (5)

$H_j = \frac{c_{j,n+1-j}}{d_{j+1,n+1-j}}$ j = 1,2,3,………………………..n

Further the end elements of all the odd rows can be written directly as

$d_{1,0+1} = d_{2,n} = d_{3,n-1} = \ldots\ldots = d_{n+1,1} = 1$

to obtain the reduced model of order m is obtained by truncating equation (3) after the first 2m terms and the transfer funtion of reduced model is obtain by inverting the equation (5) and constructing the inversion table as follows.

d_{11}
c_{11}
d_{21} 1
c_{21} c_{22}
……………………………….
……………………………….

$d_{m,1}$ $d_{m,2}$……………………….. $d_{m,m-1}$ 1
$c_{m,1}$ $c_{m,2}$………………………… $c_{m,m-1}$ $c_{m,m}$ (6)
$d_{m+1,1}$ $d_{m+1,2}$……………………… $d_{m+1,m}$ 1

$H_m = \frac{d_{11}}{c_{11}}$ $h_m = \frac{c_{11}}{d_{21}}$

$H_{m-1} = \frac{1}{c_{22}}$ $h_{m-1} = \frac{d_{21}}{c_{31}}$

………………………………………
………………………………………

$H_1 = \frac{1}{c_{m,m}}$ $h_1 = \frac{c_{m,1}}{d_{m+1,1}}$

Starting with $d_{11}=1$, the elements of the 2nd , third and subsequent rows are calculated recursively from the following

$d_{j+1,1} = h_{m+1-j}\, c_{j,1}$ j = 1,2,3………………….. m
$d_{j,k} = d_{j-1,k-1}\, h_{m+2-j}\, c_{j-1,k}$ k = 2,3…………………… j-1
$c_{j,k} = c_{j-1k+} H_{m+1-j}\, d_{j,k}$ j = 2,3……………………. m
k = 1,2……………… j-1

the end element can be written by inspection according as
as

$d_{jj} = 1$ j = 1,2,………. m+1 (7)
$c_{jj}=H_{m+1-j}$ j = 1,2,……….. m

There are (2m+1) rows in complete array. The denominator $D_m(s)$ can be written from (2m+1)th row of the array

$D_m(s) = s^m + d_{m+1,m}S^{m-1} + \ldots\ldots + d_{m+1,2}S + d_{m+1,1}$

Step- 2 Determination of the numerator coefficients of the reduced model by using the pade approximation the original nth order system can be expanded in power series about s=0 as

$G_n(s) = \frac{\sum_0^{n-1} c_{1,1} s}{\sum_0^{n} d_{1,i} s} = e_0 + e_1 s + e_2 s^2 + \ldots\ldots$

The coefficients of the power series expansion can also be calculated as follow

$e_0 = \frac{c_{,1,1}}{d1,1}$

$ei = \frac{1}{d_{1,1}}\left[a_i - \sum_{j=a} a_j e_{i-j}\right]$ i>0 (8)

$a_i = 0$ i>n-1

The kth Order reduced model is taken as

$$R_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} c_i s^1}{\sum_{i=0}^{k} d_i s^1} \quad (9)$$

Here $D_k(s)$ is known through equations (3-7)
For $N_k(s)$ of equation (9) to be pade approximations of $G_n(s)$ of equation (8) we have
$C_0 = d_0e_0$
$C_1 = d_0e_1 + d_1c_0$
……………………….
…………………………….
$C_{k-1} = d_0e_{k-1} + d_1e_{k-2} + \ldots\ldots + d_{k-2}e_1 + b_{k-1}e_0$

The coefficients c_j; j=0,1,2,…….. k-1 can found by solving the above k linear equatons hence the numerator $N_k(s)$ is obtained as
$N_k(s) = c_0 + c_1s + c_2s^2 + c_3s^3 + \ldots\ldots\ldots + c_{k-1}s^{k-1}$

IV. METHOD FOR COMPARISON

In order to check the accuracy of the proposed method the relative integral square error ISE index in between the transient parts of the reduced models using MATLAB/Simulink. The relative integral square error RISE is defined as.

$$ISE = \int_0^{\infty} [y(t) - y_m(t)]^2 \, dt$$

V. NUMERICAL EXAMPLES

The proposed method explains by considering numerical example, taken from the literature. The goodness of the proposed method is measured by calculating integral square error (ISE) between the transient parts of the original and reduced model using MATLAB. The ISE should be minimum for better approximation i.e close the $R_m(s)$ to $G_n(s)$, Which is given by

$$ISE = \int_0^{\infty} [y(t) - y_m(t)]^2 \, dt$$

Where y(t) and $y_m(t)$ are the unit step respnces of original and reduced system respectively.

Example:- Consider a 4th order system from the literature.

$$G_4(s) = \frac{24+24s+7s^2+s^3}{24+50s+35s^2+10s^3+s^4}$$

The numerator of reduced order model $R_m(s)$ can be evaluate using equations, 4, 5, & 6 for m=2 make modified Routh array & evaluated quotients h_1, h_2, h_3 and H_1, H_2, H_3
Modified Routh Array

24	50	35	10	1
24	24	7	1	
26	28	9	1	
-2	-4	-2		
-24	-17	1		
-50	-38			
1.24	1			
-2.88				
1				

$h_1 = \frac{24}{24} = 1$ $\qquad H_1 = \frac{1}{1} = 1$
$h_2 = \frac{26}{-2} = -13$ $\qquad H_2 = \frac{-2}{1} = -2$
$h_3 = \frac{-24}{-50} = 0.48$ $\qquad H_3 = \frac{-38}{1} = -38$
$h_4 = \frac{1.24}{-2.88} = -0.43$ $\qquad H_4 = \frac{-2.88}{1} = -2.88$

with the knowledge of the first four quotients (m=2) $h_1 = 1$, $H_1 = 1$, $h_2 = -13$, $H_2 = -2$ and with the help of equation (6) construct the inversion table as follows

Inversion Table.

1		
-2		
26	1	
24	1	
24	27	1

Hence the denominator D2(s) of reduced order model $R_2(s)$ is obtained as

$D_2(s) = s^2+27s+27$

Using the equation (8-10) the following coefficients are calculated

$e_0 = 1$ $a_0 = 24$

$e_1 = -1.083333s$ $a_1 = 27s-26s = 1s$

Thus the Reduced numerator is given as- $N_2(s) = 24 + s$

Thus the Reduced Model is given as :-

$$R_2(s) = \frac{24+s}{24+27s+s^2}$$

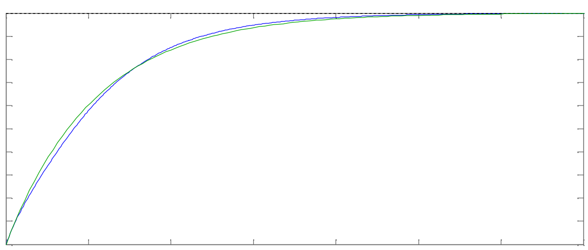

Figure1. Step Response Comparison between original system and reduced system

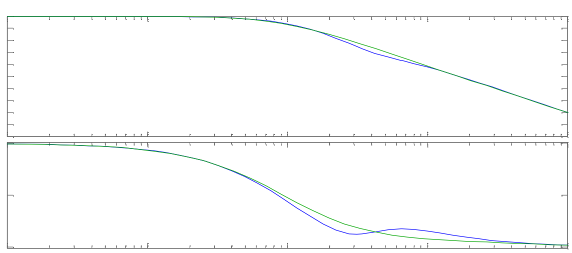

Figure 2. Bode Plots of original system and reduced system

Table – 1 Comparison of the Reduction Method

Reduction Methods	Reduction Models	ISE
Proposed Method	$R_2(S) = \frac{24 + 1s}{24 + 27S + S^2}$	0.00110
G. Parmar et al.[12] .	$R_2(s) = \frac{8 + 24.11429 s}{8 + 9s + s^2}$	0.04809
Mukherjeet et al.[13] .	$R_2(S) = \frac{4.457 + 11.39.9 s}{4.4357+4.2122s+s^2}$	0.05697
Mittal et al. [14] .	$R_2(S) = \frac{1.9906 + 7.0908 s}{2 + 3s + s^2}$	0.2689

VI. CONCLUSIONS

The authors propose an **order reduction technique** for linear, single-input single-output (SISO) high-order systems. In this method, the **denominator polynomial** of the reduced-order model is derived using the **Modified Cauer Form**, while the **numerator coefficients** are determined through the **Padé approximation technique**.

The primary advantages of the proposed method include:

- **Stability** of the reduced model,
- **Simplicity** in computation,
- **Efficiency** in implementation, and
- Suitability for **computer-based automation**.

The algorithm has been demonstrated using a benchmark example taken from the literature. The **step responses** and **Bode plots** for both the original high-order system and its second-order reduced model are shown in **Figure 1** and **Figure 2**, respectively.
A comparative analysis of the proposed method with several well-established order reduction techniques is provided in **Table I**. The results indicate that the proposed approach offers **comparable or superior performance** in terms of accuracy and computational efficiency.

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